The Finite Time Ruin Probability with the Same Heavy-tailed Insurance and Financial Risks

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Abstract

This note complements a recent study in ruin theory with risky investment by establishing the same asymptotic estimate for the finite time ruin probability under a weaker restriction on the financial risks. In particular, our result applies to a critical case that the insurance and financial risks have Pareto-type tails with the same regular index.

Key words: Asymptotics; heavy tails; finite time ruin probability.

1 The Model

Consider the following discrete time stochastic risk model. In this model, we adopt the following hypotheses: (P1) the yearly net incomes A_n, $n = 1, 2, \ldots$, constitute a sequence of independent and identically distributed (i.i.d.) random variables, (P2) the reserve is currently invested into a risky asset, which may lead to a negative return R_n at year n, and R_n, $n = 1, 2, \ldots$, also constitute a sequence of i.i.d. random variables, and (P3) the two sequences $\{A_n : n = 1, 2, \ldots\}$ and $\{R_n : n = 1, 2, \ldots\}$ are independent.

Let the initial surplus of the insurance company be $S_0 = x \geq 0$ and let S_n be the surplus of the company accumulated till the end of year n, $n = 1, 2, \ldots$. If we assume that the income A_n is calculated at the end of year n, then S_n, $n = 0, 1, \ldots$, can be characterized by the recurrence equation

$$S_n = (1 + R_n)S_{n-1} + A_n, \qquad n = 1, 2, \ldots. \tag{1}$$

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The ruin probability by the end of year n is defined by

$$\psi(x,n) = \Pr\left(\min_{0\le k\le n} S_k < 0 \;\middle|\; S_0 = x \right), \qquad n = 0, 1, \ldots.$$

For $n = 1, 2, \ldots$, write

$$X_n = -A_n, \qquad Y_n = \frac{1}{1+R_n},$$

where X_n represents the net loss during year n and Y_n represents the discount factor from year n to year $n-1$. According to Tang and Tsitsiashvili (2003), we call their generic random variables X and Y as the insurance risk and financial risk, respectively. It is reasonable to assume that $\Pr(0 < Y < \infty) = 1$.

Nyrhinen (1999) employed large deviation techniques and obtained a rough estimate for the infinite time ruin probability of the discrete time model above. Lately, Nyrhinen (2001) further extended the result to a more general stochastic model.

Define a Markov chain $\{V_n, n = 0, 1, \ldots\}$ by the recurrence equation

$$V_0 = 0, \quad V_n = Y_n \max\{0,\ X_n + V_{n-1}\}, \quad n = 1, 2, \ldots. \tag{2}$$

Tang and Tsitsiashvili (2003, Theorem 2.1) proved that

$$\psi(x,n) = \Pr(V_n > x), \qquad n = 0, 1, \ldots. \tag{3}$$

Starting from (2) and (3), Tang and Tsitsiashvili (2003) applied the mathematical induction device and obtained several precise estimates for the finite time ruin probability.

2 The Main Result

In order to state the main result, we need recall some preliminaries of heavy-tailed distributions. Two of the most important classes of heavy-tailed distributions are the class $\mathcal{L}$ and the class $\mathcal{D}$. By definition, a distribution F belongs to the class $\mathcal{L}$ if its tail $\overline{F} = 1 - F$ satisfies

$$\lim_{x\to\infty} \frac{\overline{F}(x+1)}{\overline{F}(x)} = 1;$$

F belongs to the class $\mathcal{D}$ if its tail $\overline{F}$ satisfies

$$\limsup_{x\to\infty} \frac{\overline{F}(x/2)}{\overline{F}(x)} < \infty.$$

A useful subclass of the intersection $\mathcal{L} \cap \mathcal{D}$ is $\mathcal{R}$, the class of distributions with regularly varying tails. By definition, a distribution F belongs to the class $\mathcal{R}$ if there exists some

$0 \leq \alpha < \infty$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \qquad \text{for any } y > 0.$$

We denote by $F \in \mathcal{R}_{-\alpha}$ the regularity property above. A good reference in heavy-tailed distributions with applications to insurance and finance is Embrechts et al. (1997).

Throughout, all limiting relationships are for $x \to \infty$; for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$, write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$, and write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$.

Let us go back to the model introduced above. Hereafter, denote by F and G the distributions of the insurance risk X and the financial risk Y, and denote by H the distribution of the product XY.

Tang and Tsitsiashvili (2003, Theorem 5.1) established several precise estimates for the finite time ruin probability $\psi(x, n)$ for the case that the tail of the insurance risk X is heavier than that of the financial risk Y and for the inverse case that the tail of Y is heavier than that of X. More specifically, they proved that, if $F \in \mathcal{L} \cap \mathcal{D}$ and $\mathrm{E}Y^p < \infty$ for some positive number p larger than the upper Matuszewska index of the distribution F, the relation

$$\psi(x, n) \sim \sum_{k=1}^{n} \Pr\left(X \prod_{i=1}^{k} Y_i > x\right) \tag{4}$$

holds for each $n = 1, 2, \ldots$.

In this paper we use a simpler method to prove the same result under a weaker assumption on the insurance risk Y.

Theorem 2.1. *Consider the risk model in Section 1. If $F \in \mathcal{L} \cap \mathcal{D}$ and $\overline{G}(x) = o\left(\overline{H}(x)\right)$, then (4) holds for each $n = 1, 2, \ldots$.*

Clearly, the condition $\overline{G}(x) = o\left(\overline{H}(x)\right)$ is implied by $\overline{G}(x) = o\left(\overline{F}(x)\right)$. Hence, in view of Lemma 3.5 of Tang and Tsitsiashvili (2003), our Theorem 2.1 improves Theorem 5.1 of Tang and Tsitsiashvili (2003).

Denote by $X_+ = \max\{X, 0\}$ the positive part of X. A special case of Theorem 2.1 is given below, which complements the study of Tang and Tsitsiashvili (2003) with a critical case that the insurance and financial risks have Pareto-type tails with the same regular index.

Corollary 2.1. *Consider the risk model in Section 1. If for some $\alpha > 0$, $F \in \mathcal{R}_{-\alpha}$, $G \in \mathcal{R}_{-\alpha}$, and $\mathrm{E}X_+^{\alpha} = \infty$, then relation (4) holds for each $n = 1, 2, \ldots$.*

Proof. By Theorem 2.1, it suffices to verify the condition $\overline{G}(x) = o\left(\overline{H}(x)\right)$. Actually, by Fatou's lemma we have

$$\liminf_{x\to\infty} \frac{\overline{H}(x)}{\overline{G}(x)} \geq \int_0^\infty \liminf_{x\to\infty} \frac{\overline{G}(x/y)}{\overline{G}(x)} \mathrm{d}F(y) = \mathrm{E}X_+^\alpha = \infty. \tag{5}$$

Hence, $\overline{G}(x) = o\left(\overline{H}(x)\right)$.

3 Proof of The Main Result

In order to prove Theorem 2.1, we need two lemmas. The first lemma below is from Cai and Tang (2004, Theorem 2.1):

Lemma 3.1. *If $F \in \mathcal{L}\cap\mathcal{D}$ and $G \in \mathcal{L}\cap\mathcal{D}$, then $F * G \in \mathcal{L}\cap\mathcal{D}$ and*

$$\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x).$$

The following lemma is interesting on its own right:

Lemma 3.2. *Let X and Y be two independent random variables with distributions F and G, respectively, and let H be the distribution of XY. If $F \in \mathcal{L}\cap\mathcal{D}$, $G(0) = 0$, and $\overline{G}(x) = o\left(\overline{H}(x)\right)$, then $H \in \mathcal{L}\cap\mathcal{D}$ and there exists some positive function $a(\cdot)$ satisfying $a(x) \to \infty$, $a(x) = o(x)$, and $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$.*

Proof. By Theorem 3.3(ii) of Cline and Samorodnitsky (1994), $H \in \mathcal{D}$. Thus, for an arbitrarily fixed $0 < l < 1$ and all $n = 1, 2, \ldots$,

$$\limsup_{x\to\infty} \frac{\overline{G}(l^n x)}{\overline{H}(x)} \leq \limsup_{x\to\infty} \frac{\overline{G}(l^n x)}{\overline{H}(l^n x)} \limsup_{x\to\infty} \frac{\overline{H}(l^n x)}{\overline{H}(x)} = 0.$$

Therefore, there exists an increasing sequence of numbers $v_n > nl^{-n}$, $n = 1, 2, \ldots$, such that the inequality

$$\frac{\overline{G}(l^n x)}{\overline{H}(x)} \leq \frac{1}{n}$$

holds whenever $x \geq v_n$. Define

$$a(x) = \sum_{n=1}^\infty l^n x 1_{(v_n \leq x < v_{n+1})}.$$

Obviously, $a(x) \to \infty$, $a(x) = o(x)$, and $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$. Finally, by Theorem 2.2(iii) of Cline and Samorodnitsky (1994), we know $H \in \mathcal{L}$.

Proof of Theorem 2.1:

By (3), it is trivial that the asymptotic result (4) holds for $n = 1$. Furthermore, by Lemma 3.2, the distribution of V_1 defined in (2) belongs to the class $\mathcal{L} \cap \mathcal{D}$. It follows from Lemma 3.1 that

$$\Pr\left(X_2 + V_1 > x\right) \sim \Pr\left(X_2 > x\right) + \Pr\left(X_1 Y_1 > x\right).$$

and that the distribution of the sum $X_2 + V_1$ belongs to the class $\mathcal{L} \cap \mathcal{D}$. Hence by Lemma 3.2, the distribution of V_2 defined in (2) also belongs to $\mathcal{L} \cap \mathcal{D}$. Under the assumptions of Theorem 2.1, by Lemma 3.2, there exists some positive function $a(\cdot)$ satisfying $a(x) \to \infty$, $a(x) = o(x)$, and $\overline{G}(a(x)) = o\left(\overline{H}(x)\right)$. Thus by (3) once again, we derive that

$$\begin{aligned}
\psi(x, 2) &= \int_0^{a(x)} \Pr\left(X_2 + V_1 > x/t\right) \mathrm{d}G\left(t\right) + \int_{a(x)}^{\infty} \Pr\left(X_2 + V_1 > x/t\right) \mathrm{d}G\left(t\right) \\
&= (1 + o(1)) \int_0^{a(x)} \left(\Pr\left(X_2 > x/t\right) + \Pr\left(V_1 > x/t\right)\right) \mathrm{d}G\left(t\right) + O\left(\overline{G}(a(x))\right) \\
&= (1 + o(1)) \int_0^{\infty} \left(\Pr\left(X_2 > x/t\right) + \Pr\left(V_1 > x/t\right)\right) \mathrm{d}G\left(t\right) + O\left(\overline{G}(a(x))\right) \\
&= (1 + o(1)) \left(\Pr\left(X_2 Y_1 > x\right) + \Pr\left(X_1 Y_1 Y_2 > x\right)\right).
\end{aligned}$$

This proves that relation (4) holds for $n = 2$. By the mathematical induction device we conclude that (4) holds for each $n = 1, 2, \ldots$.

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