# Local asymptotic behavior of the survival probability of the equilibrium renewal model with heavy Tails*

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**Abstract**

Recently, Tang[1] established a local asymptotic relation for the ruin probability in the Cramér-Lundberg risk model. In this short note we extend the corresponding result to the equilibrium renewal risk model.

**Key words:** geometric sums, heavy-tailed distribution, ladder height, the equilibrium renewal model, the ruin probability.

## 1 Introduction and the models

Throughout, for a given non-negative random variable (r.v.) $X$ with a distribution function (d.f.) $F(x) = \mathbb{P}(X \le x)$ and a finite mean $\mu > 0$, its tail is denoted by $\overline{F}(x) = F(x,\infty) = 1 - F(x)$ and its equilibrium distribution by $F_e(x) = \mu^{-1}\int_0^x \overline{F}(y)\mathrm{d}y$ for $x \ge 0$. For any $0 < a < b < \infty$ the integral symbol $\int_a^b$ is understood as $\int_{(a,b]}$, but $\int_a^\infty = \int_{(a,\infty)}$ and $\int_0^b = \int_{[0,b]}$. All limit relationships are for $x \to \infty$ unless stated otherwise.

The renewal model is one of the basic risk models in insurance and finance. As summarized by Embrechts *et al.*[2], it has the following structure:

(a) The claim sizes $Z_i$, $i \ge 1$, form a sequence of independent, identically distributed (i.i.d.), and non-negative r.v.'s with common d.f. $F$ and finite mean $\mu$;

(b) The claim inter-arrival times $\theta_i$, $i \ge 1$, are also i.i.d. non-negative r.v.'s with common finite mean $m > 0$, independent of the sequence $\{Z_i, i \ge 1\}$;

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(c) The number of claims in the interval $[0, t]$ is denoted by

$$N(t) = \sup\{n \geq 1 : T_n \leq t\}, \qquad t \geq 0,$$

where $T_n = \sum_{i=1}^{n} \theta_i$ and $\sup\phi = 0$ by convention;

(d) The loss process is then defined as

$$S(t) = \sum_{i=1}^{N(t)} Z_i - ct, \qquad t \geq 0,$$

where the constant $0 < c < \infty$ denotes the premium rate.

If in (b), the inter-arrival times $\theta_i$, $i \geq 2$, are i.i.d. r.v.'s with common d.f. $G$ and finite mean $m$, but $\theta_1$ has a possibly different d.f. $G_1$, the model above is called the delayed renewal model. Especially, if the d.f. $G_1$ coincides with the equilibrium d.f. of $G$, that is,

$$G_1(x) = G_e(x) = \frac{1}{m} \int_0^x \overline{G}(y) \mathrm{d}y, \qquad x \geq 0,$$

then the model above is called the equilibrium renewal model. If in (b) the inter-arrival times $\theta_i$, $i \geq 1$, are i.i.d. and exponentially distributed, the model above is called the Cramér-Lundberg model. See [3, 4, 5] for details; see also [6] for some recent study of the equilibrium renewal model.

Throughout the paper we assume that

$$\rho = \frac{c\mathbb{E}\theta_2 - \mathbb{E}Z_2}{\mathbb{E}Z_2} = \frac{cm - \mu}{\mu} > 0, \tag{1.1}$$

which can be interpreted as the safety loading condition. Under this condition, by the law of large numbers we know that the ultimate maximum $M$ of the loss process $S(t)$ is a.s. finite. Denote the d.f. of the maximum $M$ by $R$, which is often called survival probability. The ruin probability $\psi(x)$ is then defined by

$$\psi(x) = R(x, \infty) = \mathbb{P}(M > x) \qquad \text{for } x \geq 0,$$

where we denote $R(x, x + z] = R(x + z) - R(x)$. This paper aims at a local asymptotic relation for the survival probability $R$; see Theorem 3.1 below, which extends the main result of Tang[1] from the Cramér-Lundberg model to the equilibrium renewal model.

# 2 Heavy-tailed distributions

Heavy-tailed distributions have been the focus of interest of many researchers in the recent literature of insurance and finance; see [2, 7], among others. We list here some of the most important classes of heavy-tailed distributions, where the d.f. $F$ involved is always assumed to be supported on $[0, \infty)$:

1. the class $\mathcal{S}$ (*Subexponential*): a d.f. $F$ belongs to $\mathcal{S}$ iff

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \qquad \text{for any } n \geq 2 \text{ (or, equivalently, for some } n \geq 2), \tag{2.1}$$

   where $F^{*n}$ denotes the $n$-fold convolution of $F$, with convention that $F^{*0}$ is a d.f. degenerate at 0;

2. the class $\mathcal{L}$ (*Long-tailed*): a d.f. $F$ belongs to $\mathcal{L}$ iff

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \qquad \text{for any } y \text{ (or, equivalently, for } y = 1); \tag{2.2}$$

3. the class $\mathcal{S}^*$: a d.f. $F$ with a finite mean $\mu > 0$ belongs to $\mathcal{S}^*$ iff

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \sim 2\mu\overline{F}(x).$$

The class $\mathcal{S}^*$ was first introduced by Klüppelberg [8]. Till now it has been used in many studies of applied probability. The following inclusions are well-known; see [8]:

$$\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L}. \tag{2.3}$$

For a d.f. $F \in \mathcal{S}^*$, from the discussion of Tang[1] it holds that

$$\lim_{A\to\infty} \lim_{x\to\infty} \frac{\int_A^{x-A} \overline{F}(x-y)\overline{F}(y)\mathrm{d}y}{\overline{F}(x)} = 0. \tag{2.4}$$

We will apply these results in the sequel.

# 3 The main result

The following asymptotic relation for the ruin probability is well-known: in the renewal model with the safety loading condition (1.1), if $F_e \in \mathcal{S}$, then

$$R(x, \infty) \sim \rho^{-1}\overline{F_e}(x); \tag{3.1}$$

see [9]. Kong *et al.*[10] extended this result to the equilibrium renewal model. Recently, Tang[11, 1] established a corresponding local asymptotic relation in the Cramér-Lundberg model and gave some motivations for this study. Further investigations in this direction can be found in [12, 13, 14].

In this paper we successfully extend Tang's result from the Cramér-Lundberg model to the equilibrium renewal model:

**Theorem 3.1** Consider the equilibrium renewal model with the safety loading condition (1.1). If the claimsize has a non-lattice d.f. $F \in \mathcal{S}^*$, then we have

$$R(x, x+z] \sim \frac{z}{\rho\mu}\overline{F}(x) \qquad \text{for any } z > 0. \tag{3.2}$$

# 4 Proof of Theorem 1

## 4.1 Ladder heights

Tang[11] deeply investigated the distributions of the ascending and descending ladder heights of a random walk with negative drift by means of the sophisticated Wiener-Hopf theory. For the sake of self-containedness we borrow some results from [11] to the present study.

First of all we introduce some important characteristics of the equilibrium renewal model:

1. the ascending ladder epochs

$$\pi_1 = \inf\{t : S(t) > 0\}, \quad \pi_n = \inf\{t : S(t) > S(\pi_{n-1})\}, \qquad n \geq 2;$$

2. the ascending ladder heights

$$L_1 = S(\pi_1), \quad L_n = S(\pi_n) - S(\pi_{n-1}), \qquad n \geq 2;$$

3. the descending ladder epochs

$$\pi_1^- = \inf\{T_n : S(T_n) < 0\}, \quad \pi_n^- = \inf\{T_n : S(T_n) < S(\pi_{n-1}^-)\}, \qquad n \geq 2;$$

4. the descending ladder heights

$$L_1^- = -S(\pi_1^-), \quad L_n^- = S(\pi_{n-1}^-) - S(\pi_n^-), \qquad n \geq 2.$$

In case the ascending ladder epoch $\pi_{n-1} = \infty$, we define that $\pi_n = L_n = \infty$. Clearly, $\{L_n | (\pi_{n-1} < \infty), n \geq 1\}$ is a sequence of independent positive r.v.'s, and $\{L_n | (\pi_{n-1} < \infty), n \geq 2\}$ are identically distributed. Let $H_1$ be the d.f. of the ladder $L_1$, and $H$ be the d.f. of the ladder $L_2$. The following results related to the d.f.'s $H_1$ and $H$ in the equilibrium renewal model are well-known (see [3, 11, 15, 16]):

$$H_1(x, \infty) = \frac{\mu}{cm} \overline{F_e}(x), \qquad x \geq 0; \qquad H(x, \infty) \sim \frac{\mu}{\mathbb{E}L_2^-} \overline{F_e}(x); \tag{4.1}$$

$$q_1 = \mathbb{P}(L_1 = \infty) = \frac{cm - \mu}{cm} > 0; \qquad q = \mathbb{P}(L = \infty) = \frac{cm - \mu}{\mathbb{E}L_2^-} > 0. \tag{4.2}$$

The latter two formulae above indicate that the d.f.'s $H_1$ and $H$ are defective with deficits $q_1$ and $q$, respectively. We further denote the standard version of $H$ by

$$H_s(x) = \frac{H(x)}{1-q}, \qquad 0 < x < \infty.$$

## 4.2 Two lemmas

Before giving the proof of Theorem 3.1, we prepare two lemmas as follows.

**Lemma 4.1** Consider the equilibrium renewal model with the safety loading condition (1.1). If the claimsize has a non-lattice d.f. $F \in \mathcal{S}^*$, then we have that, for any $z > 0$ and any $n \in \mathbb{N}$,

$$\left(F_e * H_s^{*(n-1)}\right)(x, x+z] \sim \left(\frac{n-1}{(1-q)\mathbb{E}L_2^-} + \frac{1}{\mu}\right) z\overline{F}(x). \tag{4.3}$$

*Proof.* Clearly, Lemma 3 of Asmussen *et al.* [12] indicates that

$$H_s^{*(n-1)}(x, x+z] \sim \frac{(n-1)z}{(1-q)\mathbb{E}L_2^-}\mathbb{P}\left(Z_2 - c\theta_2 > x\right) \sim \frac{(n-1)z}{(1-q)\mathbb{E}L_2^-}\overline{F}(x),$$

where the last step can be proved by $F \in \mathcal{L}$ and the dominated convergence theorem, as

$$\mathbb{P}\left(Z_2 - c\theta_2 > x\right) = \overline{F}\left(x\right)\int_0^\infty \frac{\overline{F}\left(x+ct\right)}{\overline{F}\left(x\right)}\mathrm{d}G(t) \sim \overline{F}\left(x\right).$$

Then by Lemma 1 of [1] we obtain the proof of Lemma 4.1. □

**Lemma 4.2** Consider the equilibrium renewal model with the safety loading condition (1.1). If the claimsize has a non-lattice d.f. $F \in \mathcal{S}^*$, then there is some $C > 0$ such that for any $z > 0$ and any $n \in \mathbb{N}$,

$$\left(F_e * H_s^{*(n-1)}\right)(x, x+z] \le C(1+\varepsilon)^n\overline{F}(x). \tag{4.4}$$

*Proof.* Lemma 2 of [12] indicates that there is some $C_1 > 0$ such that for any $z > 0$ and any $n \in \mathbb{N}$,

$$H_s^{*(n-1)}(x, x+z] \le C_1(1+\varepsilon)^{n-1}\mathbb{P}\left(Z_2 - c\theta_2 > x\right) \le C_1(1+\varepsilon)^{n-1}\overline{F}(x). \tag{4.5}$$

Now we prove (4.4) by applying some techniques developed in the proof of Lemma 3 of [1]. As was done there, relation (2.4) implies that there is a sufficiently large $A > 0$ such that

$$\int_A^{x-A} \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \le \frac{\varepsilon}{2}\mu\overline{F}(x) \tag{4.6}$$

holds for all $x > 0$. For this fixed $A$ we can find some sufficiently large $x_2 > A$ such that

$$\overline{F}(x-A) \le (1+\varepsilon/2)\overline{F}(x) \tag{4.7}$$

holds for all $x \ge x_2$ since $F \in \mathcal{L}$ (recall the inclusions in (2.3)). We divide the probability on the left-hand side of (4.4) into three parts as

$$\begin{aligned}\left(F_e * H_s^{*(n-1)}\right)(x, x+z] &= \left(\int_0^A + \int_A^{x-A} + \int_{x-A}^{x+z}\right) H_s^{*(n-1)}(x-y, x+z-y]\mathrm{d}F_e(y) \\ &= J_1 + J_2 + J_3.\end{aligned}$$

First we deal with $J_1$. Clearly, if $x \le x_2$ then

$$J_1 \le 1 \le (\overline{F}(x_2))^{-1}\overline{F}\,(x)\,,$$

and if $x > x_2$ then by (4.5) and (4.7) we have

$$\begin{aligned} J_1 &\le C_1(1+\varepsilon)^{n-1}\int_0^A \overline{F}(x-y)\mathrm{d}F_e(y) \\ &\le C_1(1+\varepsilon)^{n-1}\overline{F}(x-A) \\ &\le C_1(1+\varepsilon)^{n-1}(1+\varepsilon/2)\overline{F}(x). \end{aligned}$$

Thus in any case we obtain

$$J_1 \le \left((\overline{F}(x_2))^{-1} + C_1(1+\varepsilon)^{n-1}(1+\varepsilon/2)\right)\overline{F}(x). \tag{4.8}$$

As for $J_2$, from (4.5) and (4.6), we have

$$J_2 \le C_1(1+\varepsilon)^{n-1}\int_A^{x-A}\overline{F}(x-y)\mathrm{d}F_e(y) \le \frac{C_1(1+\varepsilon)^{n-1}\varepsilon}{2}\overline{F}(x). \tag{4.9}$$

Finally we turn to $J_3$. By integration by parts we derive

$$\begin{aligned} J_3 &= -H_s^{*(n-1)}(A, A+z]F_e(x-A) + \int_0^{A+z} F_e(x+z-y)\mathrm{d}H_s^{*(n-1)}(y) \\ &\quad - \int_0^A F_e(x-y)\mathrm{d}H_s^{*(n-1)}(y) \\ &\le H_s^{*(n-1)}(A, A+z]F_e(x-A, x+z-A] + \int_0^A F_e(x-y, x+z-y]\mathrm{d}H_s^{*(n-1)}(y) \\ &\le \frac{z}{\mu}\overline{F}(x-A) + \frac{z}{\mu}\int_0^A \overline{F}(x-y)\mathrm{d}H_s^{*(n-1)}(y) \\ &\le \frac{2z}{\mu}\overline{F}(x-A) \le \frac{2z}{\mu}D_1\overline{F}(x), \end{aligned} \tag{4.10}$$

where

$$D_1 = \sup_{x\ge 0}\frac{\overline{F}(x-A)}{\overline{F}(x)} < \infty$$

since $F \in \mathcal{L}$. Combination of (4.8), (4.9) with (4.10) yields the result (4.4) with a suitable constant coefficient $C$. □

## 4.3 Proof of Theorem 3.1

Now we are ready to prove the main result. We introduce a stopping time $\nu = \min\{n \ge 1 : \pi_n = \infty\}$. It is easy to see that

$$\mathbb{P}(\nu = 1) = q_1, \qquad \mathbb{P}(\nu = n) = q(1-q_1)(1-q)^{n-1}, \qquad n \ge 2.$$

The maximum $M$ of the risk process $S(t)$ can be rewritten to[11, 17]

$$M = \sum_{n=0}^{\nu-1} L_n,$$

where $L_0 = 0$ by convention. Therefore, by the same method as applied in Lemma 4.1 we obtain that

$$\begin{aligned} R(x, x+z] &= \sum_{n=1}^{\infty} P\left(x < \sum_{k=1}^{n} L_k \leq x+z, L_{n+1} = \infty\right) \\ &= q \sum_{n=1}^{\infty} \left(H_1 * H^{*(n-1)}\right)(x, x+z] \\ &= q \sum_{n=1}^{\infty} (1-q_1)(1-q)^{n-1} \left(F_e * H_s^{*(n-1)}\right)(x, x+z]. \end{aligned}$$

We choose $\varepsilon > 0$ in (4.4) sufficiently small such that $(1-q)(1+\varepsilon) < 1$. Then, applying the dominated convergence theorem ensured by Lemma 4.2, we obtain

$$\begin{aligned} \lim_{x\to\infty} \frac{R(x, x+z]}{\overline{F}(x)} &= q \sum_{n=1}^{\infty} (1-q_1)(1-q)^{n-1} \lim_{x\to\infty} \frac{\left(F_e * H_s^{*(n-1)}\right)(x, x+z]}{\overline{F}(x)} \\ &= q \sum_{n=1}^{\infty} (1-q_1)(1-q)^{n-1} \left(\frac{n-1}{(1-q)\mathbb{E}L_2^-} + \frac{1}{\mu}\right) z, \end{aligned}$$

where we used Lemma 4.1 in the last step. Substituting the values of $q_1$ and $q$ given in (4.2) to the above, some simple calculation leads to

$$\lim_{x\to\infty} \frac{R(x, x+z]}{\overline{F}(x)} = \frac{z}{\rho\mu}.$$

This proves the local result (3.2). □

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