

# The Strong Law of Large Numbers for Extended Negatively Dependent Random Variables

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## Abstract

A sequence of random variables is said to be extended negatively dependent (END) if the tails of its finite-dimensional distributions in the lower-left and upper-right corners are dominated by a multiple of the tails of the corresponding finite-dimensional distributions of a sequence of independent random variables with the same marginal distributions. The goal of this paper is to establish the strong law of large numbers for a sequence of END and identically distributed random variables. In doing so we derive some new inequalities of large-deviation type for the sums of END and identically distributed random variables being suitably truncated. We also show applications of our main result to risk theory and renewal theory.

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## 1 Introduction

Random variables  $X_k$ ,  $k = 1, \dots, n$ , are said to be lower extended negatively dependent (LEND) if there is some  $M > 0$  such that, for all  $x_k$ ,  $k = 1, \dots, n$ ,

$$\Pr \left( \bigcap_{k=1}^n (X_k \leq x_k) \right) \leq M \prod_{k=1}^n \Pr (X_k \leq x_k); \quad (1.1)$$

they are said to be upper extended negatively dependent (UEND) if there is some  $M > 0$  such that, for all  $x_k$ ,  $k = 1, \dots, n$ ,

$$\Pr \left( \bigcap_{k=1}^n (X_k > x_k) \right) \leq M \prod_{k=1}^n \Pr (X_k > x_k); \quad (1.2)$$

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and they are said to be extended negatively dependent (END) if they are both LEND and UEND. A sequence of infinitely many random variables  $\{X_k, k = 1, 2, \dots\}$  is said to be LEND/UEND/END if for each positive integer  $n$ , the random variables  $X_k, k = 1, \dots, n$ , are LEND/UEND/END, respectively. Sometimes we need to specifically mention the dominating constant  $M$  associated with the LEND/UEND/END structures.

When  $M = 1$ , inequalities (1.1) and (1.2) describe lower and upper negative dependence, respectively. The concept of negative dependence has been extensively investigated since it was proposed by Ebrahimi and Ghosh (1981) and Block et al. (1982). In particular, Matula (1992) established the strong law of large numbers for pairwise negatively dependent random variables. A key step of Matula's (1992) derivation is that, by Hoeffding's identity, the covariance of two negatively dependent random variables, being suitably truncated, is non-positive. Hence, the pairwise negative dependence greatly prevents the partial sums from diverging to infinite. In general, this implication is not true for END random variables. Therefore, the approach we shall employ in this paper is essentially different from Matula's (1992). Recent developments of the strong law of large numbers for negatively dependent random variables can be found in Bingham and Nili Sani (2004), Gerasimov (2009), and Baek et al. (2009), among others.

As a natural generalization of negative dependence, the concept of END was proposed by Liu (2009) and further promoted by Chen et al. (2010) in the study of precise large deviations. The END structure covers all negative dependence structures and, more interestingly, it covers certain positive dependence structures. A sufficient condition for LEND or UEND is given in Lemma 2.1 below. In particular, by this lemma, every  $n$ -dimensional Farlie-Gumbel-Morgenstern (FGM) distribution describes a specific END structure. Recall that an  $n$ -dimensional FGM distribution has the form

$$F_{1,\dots,n}(x_1, \dots, x_n) = \left( \prod_{k=1}^n F_k(x_k) \right) \left( 1 + \sum_{1 \leq i < j \leq n} a_{ij} \overline{F}_i(x_i) \overline{F}_j(x_j) \right),$$

where  $F_k = 1 - \overline{F}_k$ ,  $k = 1, \dots, n$ , are corresponding marginal distributions and  $a_{ij}$  are real numbers chosen such that  $F_{1,\dots,n}$  is a proper  $n$ -dimensional distribution. We refer the reader to Kotz et al. (2000) for a general account on multivariate FGM distributions. Due to its transparent structure and great flexibility of adjusting dependence strength, this family of multivariate distributions is often used in modelling dependent investment returns in finance or dependent claim sizes in insurance; see, for example, Tang and Vernic (2007) and Cossette et al. (2008), among many others. Hashorva (2001) established limit theorems for a sequence of random variables with FGM finite-dimensional distributions.

Our main result is the following:

**Theorem 1.1.** *Let  $\{X_k, k = 1, 2, \dots\}$  be a sequence of END random variables with common distribution  $F$ . Denote by  $S_n$  its  $n$ th partial sum,  $n = 1, 2, \dots$ . Then  $S_n/n \xrightarrow{\text{a.s.}} \mu$  as  $n \rightarrow \infty$  for some real number  $\mu$  if and only if  $E|X_1| < \infty$ , and for each case  $\mu = EX_1$ .*

The rest of this paper consists of three sections: Section 2 prepares a series of important lemmas, Section 3 proves Theorem 1.1, and Section 4 proposes two applications of Theorem 1.1 to risk theory and renewal theory, respectively.

## 2 Lemmas

Let random variables  $X_k, k = 1, \dots, n$ , follow distributions  $F_k, k = 1, \dots, n$ , respectively, and let  $C(u_1, \dots, u_n) : [0, 1]^n \mapsto [0, 1]$  be their copula, so that, by Sklar's theorem, it holds for all  $x_k, k = 1, \dots, n$ , that

$$\Pr \left( \bigcap_{k=1}^n (X_k \leq x_k) \right) = C(F_1(x_1), \dots, F_n(x_n)). \quad (2.1)$$

In particular, if  $F_k, k = 1, \dots, n$ , are continuous then the copula  $C(u_1, \dots, u_n)$  fulfilling (2.1) is unique and is identical to the joint distribution of the uniform variates  $F(X_k), k = 1, \dots, n$ . See Joe (1997) and Nelsen (2006) for comprehensive treatments on copulas.

For every nonempty subset  $I$  of  $\{1, \dots, n\}$ , the corresponding marginal copula is

$$C(u_i : i \in I) = C(u_1, \dots, u_n) \Big|_{u_j=1 \text{ for } j \in \{1, \dots, n\} \setminus I}$$

and the corresponding marginal copula density, if it exists, is equal to

$$c(u_i : i \in I) = \left( \prod_{i \in I} \frac{\partial}{\partial u_i} \right) C(u_i : i \in I).$$

Trivially, if the copula density  $c(u_1, \dots, u_n)$  is bounded over  $(u_1, \dots, u_n) \in [0, 1]^n$ , so is every marginal copula density.

Motivated by Corollary 3.1 of Ko and Tang (2008), in the following lemma we show a sufficient condition for the random variables  $X_k, k = 1, \dots, n$ , to be LEND or UEND:

**Lemma 2.1.** *Assume that random variables  $X_k, k = 1, \dots, n$ , have continuous distributions  $F_k, k = 1, \dots, n$ , respectively, and possess a copula  $C(u_1, \dots, u_n)$  with a copula density  $c(u_1, \dots, u_n)$  well defined on  $(0, 1)^n$ .*

- (a) *If every marginal copula density is bounded in a neighborhood of the origin (whose coordinates are all 0), then  $X_k, k = 1, \dots, n$ , are LEND;*
- (b) *If every marginal copula density is bounded in a neighborhood of the ultimate vertex (whose coordinates are all 1), then  $X_k, k = 1, \dots, n$ , are UEND.*

*Proof.* We only prove (a) since (b) can be proven in the same way. Let  $\varepsilon \in (0, 1)$  and  $M^* > 0$  be two constants such that every marginal copula density is bounded by  $M^*$  as long as all its arguments fall into the interval  $(0, \varepsilon]$ . Then, for arbitrarily given  $x_k, k = 1, \dots, n$ , denote by  $I$  the set of all  $i$  for which  $F_i(x_i) \leq \varepsilon$  and by  $I^c$  its complement,  $I^c = \{1, \dots, n\} \setminus I$ . When  $I$  is nonempty, we have

$$\begin{aligned}
\Pr \left( \bigcap_{k=1}^n (X_k \leq x_k) \right) &\leq \Pr \left( \bigcap_{i \in I} (X_i \leq x_i) \right) \\
&= C(F_i(x_i) : i \in I) \\
&= \int \cdots \int_{0 < u_i \leq F_i(x_i), i \in I} c(u_i : i \in I) \prod_{i \in I} du_i \\
&\leq M^* \prod_{i \in I} F_i(x_i) \\
&\leq M^* \prod_{i \in I} F_i(x_i) \prod_{j \in I^c} \frac{F_j(x_j)}{\varepsilon} \\
&\leq \frac{M^*}{\varepsilon^n} \prod_{k=1}^n F_k(x_k).
\end{aligned}$$

Clearly, if  $I$  is empty then we have

$$\Pr \left( \bigcap_{k=1}^n (X_k \leq x_k) \right) \leq 1 \leq \frac{1}{\varepsilon^n} \prod_{k=1}^n F_k(x_k).$$

Thus, inequality (1.1) holds with  $M = \varepsilon^{-n} (M^* \vee 1)$ .  $\square$

Let us collect some preliminaries regarding the concept of END for later use. Throughout this paper, for a real number  $x$  write  $x^+ = x \vee 0$  and  $x^- = (-x) \vee 0$  as the positive and negative parts of  $x$ , respectively. The following lemma is essentially a refinement of Lemma 3.1 of Liu (2009):

**Lemma 2.2.** *Let  $X_k, k = 1, \dots, n$ , be random variables and let  $g_k, k = 1, \dots, n$ , be real functions.*

(a) *If  $X_k, k = 1, \dots, n$ , are UEND with some dominating coefficient  $M > 0$ , then*

$$\mathbb{E} \left( \prod_{k=1}^n X_k^+ \right) \leq M \prod_{k=1}^n \mathbb{E} X_k^+.$$

(b) *Assume that  $X_k, k = 1, \dots, n$ , are LEND/UEND/END with some dominating constant  $M > 0$ . If  $g_k, k = 1, \dots, n$ , are all non-decreasing then  $g_k(X_k), k = 1, \dots, n$ , are still LEND/UEND/END, while if  $g_k, k = 1, \dots, n$ , are all non-increasing then  $g_k(X_k), k = 1, \dots, n$ , are UEND/LEND/END. For each case, the dominating constant  $M > 0$  remains unchanged.*

*Proof.* (a) By Fubini's theorem and inequality (1.2), we have

$$\begin{aligned}
\mathbb{E} \left( \prod_{k=1}^n X_k^+ \right) &= \mathbb{E} \left( \int \cdots \int_{(x_1, \dots, x_n) \in [0, \infty)^n} \left( \prod_{k=1}^n 1_{(x_k < X_k^+)} \right) \prod_{k=1}^n dx_k \right) \\
&= \int \cdots \int_{(x_1, \dots, x_n) \in [0, \infty)^n} \Pr \left( \bigcap_{k=1}^n (X_k > x_k) \right) \prod_{k=1}^n dx_k \\
&\leq M \int \cdots \int_{(x_1, \dots, x_n) \in [0, \infty)^n} \prod_{k=1}^n \Pr(X_k > x_k) \prod_{k=1}^n dx_k \\
&= M \prod_{k=1}^n \mathbb{E} X_k^+. \tag{2.2}
\end{aligned}$$

Note that the derivation of (2.2) is still valid even when  $\mathbb{E} X_k^+ = \infty$  for some  $k = 1, \dots, n$ .

(b) As all assertions can be proven in the same way, we only prove that if  $X_k, k = 1, \dots, n$ , are LEND with some dominating constant  $M > 0$  and  $g_k, k = 1, \dots, n$ , are all non-decreasing then  $g_k(X_k), k = 1, \dots, n$ , are still LEND with the same dominating constant  $M$ . For each  $k = 1, \dots, n$  and each real number  $y_k$ , the event  $(g_k(X_k) \leq y_k)$  is equivalent to either  $\Delta_k = (X_k \leq x_k)$  or  $\Delta_k = (X_k < x_k)$  for  $x_k = \sup\{x : g_k(x) \leq y_k\} \in [-\infty, \infty]$ . For the latter case,  $\Delta_k = (X_k < x_k)$  can be approximated by  $(X_k \leq x_k^*)$  as  $x_k^* \rightarrow x_k^-$ . Therefore, by relation (1.1) and the continuity of the probability measure, we have

$$\Pr \left( \bigcap_{k=1}^n (g_k(X_k) \leq y_k) \right) = \Pr \left( \bigcap_{k=1}^n \Delta_k \right) \leq M \prod_{k=1}^n \Pr(\Delta_k) = M \prod_{k=1}^n \Pr(g_k(X_k) \leq y_k).$$

This ends the proof of Lemma 2.2 □

The following generalized Borel-Cantelli lemma is due to Kochen and Stone (1964) and was retrieved recently by Yan (2006):

**Lemma 2.3.** *Let  $\{A_n, n = 1, 2, \dots\}$  be a sequence of events such that  $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$ . Then*

$$\Pr(A_n \text{ i.o.}) \geq \limsup_{n \rightarrow \infty} \frac{\sum_{1 \leq i < j \leq n} \Pr(A_i) \Pr(A_j)}{\sum_{1 \leq i < j \leq n} \Pr(A_i A_j)}.$$

Let  $F$  be a distribution on  $(-\infty, \infty)$ . For arbitrarily fixed  $\delta > 0$ , define auxiliary functions  $f_\delta, f_\delta^\pm$  as

$$\begin{aligned}
f_\delta(x) &= x^{-\delta} \int_{|y| \leq x} |y|^{1+\delta} F(dy) \\
&= x^{-\delta} \int_{0 \leq y \leq x} y^{1+\delta} F(dy) + x^{-\delta} \int_{-x \leq y \leq 0} (-y)^{1+\delta} F(dy) \\
&= f_\delta^+(x) + f_\delta^-(x), \quad x > 0. \tag{2.3}
\end{aligned}$$

These auxiliary functions will be crucial for establishing our key inequalities for the tail probabilities of the sums of END random variables. The following result is elementary:

**Lemma 2.4.** *For the auxiliary functions  $f_\delta$ ,  $f_\delta^\pm$  defined in (2.3), as  $x \rightarrow \infty$ ,*

- (a) *if  $x \Pr(X > x) \rightarrow 0$  then  $f_\delta^+(x) \rightarrow 0$ ;*
- (b) *if  $x \Pr(X < -x) \rightarrow 0$  then  $f_\delta^-(x) \rightarrow 0$ ;*
- (c) *if  $x \Pr(|X| > x) \rightarrow 0$  then  $f_\delta(x) = f_\delta^+(x) + f_\delta^-(x) \rightarrow 0$ .*

*Proof.* We only prove (a) since (b) can be proven in the same way and (c) is an immediate consequence of (a) and (b). By Fubini's theorem,

$$\begin{aligned} f_\delta^+(x) &= \frac{1+\delta}{x^\delta} \int_0^x \left( \int_0^y z^\delta dz \right) F(dy) \\ &= \frac{1+\delta}{x^\delta} \int_0^x \int_z^x z^\delta F(dy) dz \\ &\leq \frac{1+\delta}{x^\delta} \int_0^x z^\delta \bar{F}(z) dz. \end{aligned}$$

For every  $\varepsilon > 0$ , there is some  $z_0 > 0$  such that  $\bar{F}(z) \leq \varepsilon z^{-1}$  for all  $z > z_0$ . Thus,

$$\begin{aligned} f_\delta^+(x) &\leq \frac{1+\delta}{x^\delta} \left( \int_0^{z_0} z^\delta \bar{F}(z) dz + \varepsilon \int_{z_0}^x z^{\delta-1} dz \right) \\ &\leq \frac{1+\delta}{x^\delta} \left( \int_0^{z_0} z^\delta \bar{F}(z) dz + \frac{\varepsilon}{\delta} x^\delta \right). \end{aligned}$$

By the arbitrariness of  $\varepsilon$ , we conclude that  $f_\delta^+(x) \rightarrow 0$  as  $x \rightarrow \infty$ . □

Let  $\{X_k, k = 1, 2, \dots\}$  be a sequence of UEND random variables with common distribution  $F$  and mean zero. For arbitrarily fixed  $0 < v < 1$ , define

$$\tilde{X}_k = -vx1_{(X_k < -vx)} + X_k1_{(-vx \leq X_k \leq vx)} + vx1_{(X_k > vx)}, \quad k = 1, 2, \dots, \quad (2.4)$$

which, by Lemma 2.2(b), are still UEND random variables. Write

$$\tilde{S}_n = \sum_{k=1}^n \tilde{X}_k, \quad n = 1, 2, \dots,$$

and  $\mu_\pm = EX_1^\pm$ . Trivially,  $\mu_+ = \mu_-$  since  $EX_1 = 0$ .

We are going to establish some key inequalities of large-deviation type for the sums  $\tilde{S}_n$ ,  $n = 1, 2, \dots$ . An important feature of the following result is that it does not require  $X_1$  to have a finite moment of order higher than 1. The obtained inequality is new even for the independent case.

**Lemma 2.5.** Consider the truncated random variables defined in (2.4), where  $X_k$ ,  $k = 1, 2, \dots$ , are UEND random variables with common distribution  $F$ , mean zero, and a dominating constant  $M > 0$ . Then, for every  $v > 0$ ,  $\gamma > 0$ ,  $0 < \delta \leq 1$ , and  $0 < \theta < 1$ , there is some  $x_0 = x_0(v, \gamma, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$  and  $x \geq (\gamma n) \vee x_0$ ,

$$\Pr\left(\tilde{S}_n > x\right) \leq M \left(f_\delta^+(vx) + vx\bar{F}(vx)\right)^{(1-\theta)/v}, \quad (2.5)$$

where the auxiliary function  $f_\delta^+$  is defined in (2.3).

*Proof.* By Lemma 2.2(b), for every  $v > 0$ ,  $h > 0$ , the random variables  $h\tilde{X}_k$ ,  $k = 1, 2, \dots$ , are still UEND with the same dominating constant  $M$  unrelated to  $v$  or  $h$ . Let  $h = h(x)$  be a positive function of  $x$  whose exact form will be specified later such that  $h(x) \rightarrow 0$  as  $x \rightarrow \infty$ . By Markov's inequality and Lemma 2.2(a), we have

$$\Pr\left(\tilde{S}_n > x\right) \leq e^{-hx} \mathbb{E}e^{h\tilde{S}_n} \leq Me^{-hx} \left(\mathbb{E}e^{h\tilde{X}_1}\right)^n. \quad (2.6)$$

Now we focus on estimation of the exponential moment  $\mathbb{E}e^{h\tilde{X}_1}$ . Clearly,

$$\begin{aligned} & \mathbb{E}e^{h\tilde{X}_1} \\ &= \left(\int_{-vx \leq y \leq 0} + \int_{0 \leq y \leq vx}\right) (e^{hy} - 1) F(dy) + (e^{hvx} - 1) \bar{F}(vx) + (e^{-hvx} - 1) F(-vx - 0) + 1 \\ &\leq \left(\int_{-vx \leq y \leq 0} + \int_{0 \leq y \leq vx}\right) (e^{hy} - 1) F(dy) + (e^{hvx} - 1) \bar{F}(vx) + 1 \\ &= I_1(x) + I_2(x) + I_3(x) + 1. \end{aligned} \quad (2.7)$$

We use an idea of Tang and Yan (2002) to deal with  $I_1(x)$ . It holds for  $y \leq 0$  that

$$0 \leq \frac{e^{hy} - 1 - hy}{h} \leq y(e^{hy} - 1) \leq -y.$$

Then, by the dominated convergence theorem,

$$\lim_{x \rightarrow \infty} \frac{I_1(x)}{h} = \int_{-\infty}^0 \lim_{h \rightarrow 0^+} \frac{e^{hy} - 1 - hy}{h} F(dy) - \mu_- = -\mu_-.$$

This means that

$$I_1(x) = -\mu_- h + o_1(h), \quad (2.8)$$

where  $o_1(h)$  is a real function of  $h > 0$  satisfying  $o_1(h)/h \rightarrow 0$  as  $h \rightarrow 0^+$ . For arbitrarily fixed  $0 < \delta \leq 1$ , by the monotonicity of the function  $(e^{hy} - 1 - hy)/y^{1+\delta}$  for  $y > 0$ , we have

$$\begin{aligned} I_2(x) &\leq \int_0^{vx} \frac{e^{hy} - 1 - hy}{y^{1+\delta}} y^{1+\delta} F(dy) + \mu_+ h \\ &\leq \frac{e^{hvx} - 1}{(vx)^{1+\delta}} \int_0^{vx} y^{1+\delta} F(dy) + \mu_+ h \\ &= \frac{e^{hvx} - 1}{hvx} f_\delta^+(vx) h + \mu_+ h. \end{aligned} \quad (2.9)$$

Rewrite  $I_3(x)$  as

$$I_3(x) = \frac{e^{hvx} - 1}{hvx} (vx\bar{F}(vx)) h. \quad (2.10)$$

We specify  $h$  to

$$h = \frac{1}{vx} \ln \left( 1 + \frac{1}{f_\delta^+(vx) + vx\bar{F}(vx)} \right). \quad (2.11)$$

Note that  $h \rightarrow 0+$  is equivalent to  $x \rightarrow \infty$ . Actually, the fact  $h \rightarrow 0+$  implies  $x \rightarrow \infty$  is trivial since both  $f_\delta^+(vx)$  and  $vx\bar{F}(vx)$  are bounded for  $x > 0$ . For the other implication, with some  $x^* > 0$  such that  $F(vx^*) - F(0) > 0$ , it holds for all  $x \geq x^*$  that

$$h \leq \frac{1}{vx} \ln \left( 1 + \frac{1}{f_\delta^+(vx)} \right) \leq \frac{1}{vx} \ln \left( 1 + \frac{(vx)^\delta}{\int_0^{vx^*} y^{1+\delta} F(dy)} \right) \sim \frac{\delta \ln x}{vx}.$$

Substituting (2.8)-(2.11) into (2.7) and noticing that both  $f_\delta^+(vx)$  and  $vx\bar{F}(vx)$  converge to 0 as  $x \rightarrow \infty$ , we obtain

$$\begin{aligned} \mathbb{E}e^{h\tilde{X}_k} &\leq o_1(h) + \frac{e^{hvx} - 1}{hvx} (f_\delta^+(vx) + vx\bar{F}(vx)) h + 1 \\ &= o_1(h) + \frac{h}{\ln \left( 1 + \frac{1}{f_\delta^+(vx) + vx\bar{F}(vx)} \right)} + 1 \\ &= o_2(h) + 1, \end{aligned} \quad (2.12)$$

where  $o_2(h)$  is a real function of  $x > 0$  satisfying  $o_2(h)/h \rightarrow 0$  as  $x \rightarrow \infty$ . Substituting (2.12) into (2.6) and using the elementary inequality  $1 + z \leq e^z$  for every real number  $z$ , we have, for all  $n = 1, 2, \dots$  and  $x \geq \gamma n$ ,

$$\Pr \left( \tilde{S}_n > x \right) \leq M \exp \{ o_2(h)n - hx \} \leq M \exp \left\{ \left( \frac{|o_2(h)|}{\gamma h} - 1 \right) hx \right\}.$$

For arbitrarily fixed  $0 < \theta < 1$ , there is some large  $x_0 > 0$  such that, for all  $x \geq x_0$ ,

$$\frac{|o_2(h)|}{\gamma h} \leq \theta.$$

It follows that, for all  $n = 1, 2, \dots$  and  $x \geq (\gamma n) \vee x_0$ ,

$$\Pr \left( \tilde{S}_n > x \right) \leq M \exp \{ -(1 - \theta)hx \} = M \left( 1 + \frac{1}{f_\delta^+(vx) + vx\bar{F}(vx)} \right)^{-(1-\theta)/v},$$

yielding inequality (2.5). □

By a symmetrization procedure, it is easy to apply Lemma 2.5 to establish a similar inequality for the tail probabilities of  $|\tilde{S}_n|$ ,  $n = 1, 2, \dots$



**Lemma 2.6.** *Consider the truncated random variables defined in (2.4), where  $X_k$ ,  $k = 1, 2, \dots$ , are END random variables with common distribution  $F$ , mean zero, and a dominating constant  $M > 0$ . Then for every  $v > 0$ ,  $\gamma > 0$ ,  $0 < \delta \leq 1$ , and  $0 < \theta < 1$ , there is some  $x_0 = x_0(v, \gamma, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$  and  $x \geq (\gamma n) \vee x_0$ ,*

$$\Pr\left(\left|\tilde{S}_n\right| > x\right) \leq 2M(f_\delta(vx) + vx \Pr(|X_1| > vx))^{(1-\theta)/v},$$

where the auxiliary function  $f_\delta$  is defined in (2.3).

*Proof.* Recall the other two auxiliary functions  $f_\delta^\pm$  defined in (2.3). Note that, by Lemma 2.2(b), if the random variables  $X_k$ ,  $k = 1, 2, \dots$ , are LEND, then  $-\tilde{X}_k$ ,  $k = 1, 2, \dots$ , are UEND. Therefore, applying Lemma 2.5 to the random variables  $-\tilde{X}_k$ ,  $k = 1, 2, \dots$ , there is some  $x_0 = x_0(v, \gamma, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$  and  $x \geq (\gamma n) \vee x_0$ ,

$$\Pr\left(-\tilde{S}_n > x\right) \leq M(f_\delta^-(vx) + vx \Pr(-X_1 > vx))^{(1-\theta)/v}. \quad (2.13)$$

A simple combination of inequalities (2.5) and (2.13) yields that, for a suitably modified constant  $x_0 = x_0(v, \gamma, \delta, \theta) > 0$  and for all  $n = 1, 2, \dots$ ,  $x \geq (\gamma n) \vee x_0$ ,

$$\begin{aligned} & \Pr\left(\left|\tilde{S}_n\right| > x\right) \\ & \leq M(f_\delta^+(vx) + vx \Pr(X_1 > vx))^{(1-\theta)/v} + M(f_\delta^-(vx) + vx \Pr(X_1 < -vx))^{(1-\theta)/v} \\ & \leq 2M(f_\delta(vx) + vx \Pr(|X_1| > vx))^{(1-\theta)/v}. \end{aligned}$$

This proves the assertion of Lemma 2.6. □

The following result essentially shows that, once  $X_1$  has a finite moment of order higher than 1, by suitably choosing the values of  $\theta$  and  $v$  in Lemma 2.6, the tail probability  $\Pr\left(\left|\tilde{S}_n\right| > x\right)$  can decay to 0 faster than any power rate. Such a result is of independent interest, particularly in the study of large deviations. Lemma 2.3 of Tang (2006) gives a result of the same flavor for negatively dependent random variables.

**Corollary 2.1.** *Recall the truncated random variables defined in (2.4).*

- (a) *In addition to the conditions of Lemma 2.5, assume that  $\mathbf{E}(X_1^+)^{1+\delta} < \infty$  for some  $0 < \delta \leq 1$ . Then, for every  $v > 0$ ,  $\gamma > 0$ , and  $0 < \theta < 1$ , there is some  $K = K(v, \gamma, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$  and  $x \geq \gamma n$ ,*

$$\Pr\left(\tilde{S}_n > x\right) \leq Kx^{-\delta(1-\theta)/v}. \quad (2.14)$$

- (b) *In addition to the conditions of Lemma 2.6, assume that  $\mathbf{E}|X_1|^{1+\delta} < \infty$  for some  $0 < \delta \leq 1$ , then, for every  $v > 0$ ,  $\gamma > 0$ , and  $0 < \theta < 1$ , there is some  $K = K(v, \gamma, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$  and  $x \geq \gamma n$ ,*

$$\Pr\left(\left|\tilde{S}_n\right| > x\right) \leq Kx^{-\delta(1-\theta)/v}. \quad (2.15)$$

*Proof.* (a) Lemma 2.5 asserts that there is some  $x_0 = x_0(v, \gamma, \delta, \theta) > 0$  such that inequality (2.5) holds for all  $n = 1, 2, \dots$  and  $x \geq (\gamma n) \vee x_0$ . Using Markov's inequality on the right-hand side of (2.5), it holds for all  $n = 1, 2, \dots$  and  $x \geq (\gamma n) \vee x_0$  that

$$\begin{aligned} \Pr(\tilde{S}_n > x) &\leq M \left( (vx)^{-\delta} \int_0^{vx} y^{1+\delta} F(dy) + vx \frac{1}{(vx)^{1+\delta}} \mathbb{E}(X_1^+)^{1+\delta} \right)^{(1-\theta)/v} \\ &\leq M \left( 2v^{-\delta} \mathbb{E}(X_1^+)^{1+\delta} \right)^{(1-\theta)/v} x^{-\delta(1-\theta)/v}. \end{aligned}$$

Furthermore, it trivially holds for all  $n = 1, 2, \dots$  and  $\gamma n \leq x \leq x_0$  that

$$\Pr(\tilde{S}_n > x) \leq 1 \leq x_0^{\delta(1-\theta)/v} x^{-\delta(1-\theta)/v}.$$

Therefore, inequality (2.14) holds with

$$K = M \left( 2v^{-\delta} \mathbb{E}(X_1^+)^{1+\delta} \right)^{(1-\theta)/v} \vee x_0^{\delta(1-\theta)/v}$$

for all  $n = 1, 2, \dots$  and  $x \geq \gamma n$ .

(b) Applying Corollary 2.1(a) to the random variables  $-\tilde{X}_k$ ,  $k = 1, 2, \dots$ , there is some  $K = K(v, \gamma, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$  and  $x \geq \gamma n$ ,

$$\Pr(-\tilde{S}_n > x) \leq Kx^{-\delta(1-\theta)/v}. \quad (2.16)$$

A simple combination of inequalities (2.14) and (2.16) yields relation (2.15) with a suitably modified constant  $K > 0$ .  $\square$

## 3 Proof of Theorem 1.1

### 3.1 Proof of the Necessity Part

Assume  $S_n/n \xrightarrow{\text{a.s.}} \mu$  as  $n \rightarrow \infty$ . This condition implies that  $X_n/n \xrightarrow{\text{a.s.}} 0$  as  $n \rightarrow \infty$ , hence that both  $X_n^+/n \xrightarrow{\text{a.s.}} 0$  and  $X_n^-/n \xrightarrow{\text{a.s.}} 0$  as  $n \rightarrow \infty$ . Define  $A_n = (X_n^+ > n)$  for  $n = 1, 2, \dots$ . We have  $\Pr(A_n \text{ i.o.}) = 0$ . This implies that

$$\sum_{n=1}^{\infty} \Pr(A_n) < \infty,$$

because otherwise, by Lemma 2.3 and the pairwise UEND of  $\{X_k, k = 1, 2, \dots\}$ , we would have

$$\Pr(A_n \text{ i.o.}) \geq \limsup_{n \rightarrow \infty} \frac{\sum_{1 \leq i < j \leq n} \Pr(A_i) \Pr(A_j)}{\sum_{1 \leq i < j \leq n} \Pr(A_i A_j)} \geq \frac{1}{M}.$$

Hence,

$$\mathbb{E}X_1^+ = \sum_{n=0}^{\infty} \int_n^{n+1} \Pr(X_1 > x) dx \leq 1 + \sum_{n=1}^{\infty} \Pr(A_n) < \infty.$$

In the same way, we can prove  $\mathbb{E}X_1^- < \infty$ . Therefore,  $\mathbb{E}|X_1| < \infty$ . Finally, by the sufficiency part of Theorem 1.1 which we are to prove below, the finiteness of  $\mathbb{E}|X_1|$  implies  $S_n/n \xrightarrow{\text{a.s.}} \mathbb{E}X_1$  as  $n \rightarrow \infty$ . Hence,  $\mu = \mathbb{E}X_1$ .

### 3.2 Proof of the Sufficiency Part

Write  $S_n^{(\pm)} = \sum_{k=1}^n X_k^\pm$  for  $n = 1, 2, \dots$  and write  $\mu_\pm = \mathbb{E}X_1^\pm$  as before. Clearly, it suffices to prove that both  $S_n^{(+)} / n \xrightarrow{\text{a.s.}} \mu_+$  and  $S_n^{(-)} / n \xrightarrow{\text{a.s.}} \mu_-$  hold as  $n \rightarrow \infty$ . We only prove the former since the latter can be proven in the same way.

For arbitrarily fixed  $v > 0$  and  $n = 1, 2, \dots$ , similarly as in (2.4) we define

$$\tilde{X}_{k,n}^+ = -vn1_{(X_k^+ - \mu_+ < -vn)} + (X_k^+ - \mu_+)1_{(-vn \leq X_k^+ - \mu_+ \leq vn)} + vn1_{(X_k^+ - \mu_+ > vn)}$$

for  $k = 1, \dots, n$ . Write  $\tilde{S}_n^{(+)} = \sum_{k=1}^n \tilde{X}_{k,n}^+$ . Let  $\varepsilon > 0$  and  $\alpha > 1$  be arbitrarily fixed and, as usual, denote by  $[z]$  the largest integer that is not larger than  $z$ . By Lemma 2.6, with suitably chosen  $0 < v < 1$  and  $0 < \theta < 1$  such that  $(1 - \theta)/v = 1$ , there is some positive integer  $n_0 = n_0(v, \varepsilon, \delta, \theta)$  such that, for all  $n \geq n_0$ ,

$$\Pr \left( \left| \frac{\tilde{S}_n^{(+)}}{n} \right| > \varepsilon \right) \leq 2M \left( (v\varepsilon n)^{-\delta} \int_{|y| \leq v\varepsilon n} |y|^{1+\delta} \tilde{F}^+(dy) + v\varepsilon n \Pr (|X_1^+ - \mu_+| > v\varepsilon n) \right),$$

where  $\tilde{F}^+$  denotes the distribution of  $X_1^+ - \mu_+$ . It follows that

$$\sum_{n=1}^{\infty} \Pr \left( \left| \frac{\tilde{S}_{[\alpha^n]}^{(+)}}{[\alpha^n]} \right| > \varepsilon \right) \leq \log_\alpha n_0 + 2M(v\varepsilon\alpha^{-1})^{-\delta}\Sigma_1 + 2Mv\varepsilon\Sigma_2, \quad (3.1)$$

with

$$\Sigma_1 = \sum_{n=1}^{\infty} \alpha^{-\delta n} \int_{|y| \leq v\varepsilon\alpha^n} |y|^{1+\delta} \tilde{F}^+(dy), \quad \Sigma_2 = \sum_{n=1}^{\infty} [\alpha^n] \Pr (|X_1^+ - \mu_+| > v\varepsilon[\alpha^n]).$$

Interchanging the order of the sum and integral in  $\Sigma_1$ , we see that, for some constant  $K_1 > 0$ ,

$$\begin{aligned} \Sigma_1 &= \int_{-\infty}^{\infty} \left( \sum_{n \geq (\log_\alpha |y| - \log_\alpha v\varepsilon) \vee 1} \alpha^{-\delta n} \right) |y|^{1+\delta} \tilde{F}^+(dy) \\ &= \int_{-\infty}^{\infty} \frac{\alpha^{-\delta((\log_\alpha |y| - \log_\alpha v\varepsilon) \vee 1)}}{1 - \alpha^{-\delta}} |y|^{1+\delta} \tilde{F}^+(dy) \\ &\leq K_1 \int_{-\infty}^{\infty} |y| \tilde{F}^+(dy) \\ &= K_1 \mathbb{E} |X_1^+ - \mu_+| < \infty. \end{aligned} \quad (3.2)$$

Moreover, it holds for some constant  $K_2 > 0$  that

$$\begin{aligned} \Sigma_2 &\leq \sum_{n=1}^{\infty} \frac{[\alpha^n]}{[(\alpha - 1)\alpha^{n-1}]} \sum_{k=1}^{[(\alpha-1)\alpha^{n-1}]} \Pr (|X_1^+ - \mu_+| > v\varepsilon([\alpha^{n-1}] + k)) \\ &\leq K_2 \sum_{n=1}^{\infty} \Pr (|X_1^+ - \mu_+| > v\varepsilon n) \\ &\leq \frac{K_2}{v\varepsilon} \mathbb{E} |X_1^+ - \mu_+| < \infty, \end{aligned} \quad (3.3)$$

where the summation  $\sum_{k=1}^{[(\alpha-1)\alpha^{n-1}]}$  produces a value 0 in case  $[(\alpha-1)\alpha^{n-1}] = 0$ . Substituting (3.2) and (3.3) into (3.1) yields that

$$\sum_{n=1}^{\infty} \Pr \left( \left| \frac{\tilde{S}_{[\alpha^n]}^{(+)}}{[\alpha^n]} \right| > \varepsilon \right) < \infty.$$

Hence,

$$\begin{aligned} & \Pr \left( \left| \frac{S_{[\alpha^n]}^{(+)}}{[\alpha^n]} - \mu_+ \right| > \varepsilon \text{ i.o.} \right) \\ & \leq \Pr \left( \left| \frac{\tilde{S}_{[\alpha^n]}^{(+)}}{[\alpha^n]} \right| > \varepsilon \text{ i.o.} \right) + \Pr \left( \frac{S_{[\alpha^n]}^{(+)}}{[\alpha^n]} - \mu_+ \neq \frac{\tilde{S}_{[\alpha^n]}^{(+)}}{[\alpha^n]} \text{ i.o.} \right) \\ & \leq \lim_{m \rightarrow \infty} \Pr \left( \bigcup_{n=m}^{\infty} \left( \frac{\tilde{S}_{[\alpha^n]}^{(+)}}{[\alpha^n]} > \varepsilon \right) \right) + \lim_{m \rightarrow \infty} \Pr \left( \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{[\alpha^n]} \left( X_k^+ - \mu_+ \neq \tilde{X}_{k, [\alpha^n]}^+ \right) \right) \\ & \leq \limsup_{m \rightarrow \infty} \sum_{n=m}^{\infty} \Pr \left( \frac{\tilde{S}_{[\alpha^n]}^{(+)}}{[\alpha^n]} > \varepsilon \right) + \limsup_{m \rightarrow \infty} \sum_{n=m}^{\infty} [\alpha^n] \Pr \left( X_k - \mu_+ > v[\alpha^n] \right) \\ & = 0, \end{aligned}$$

where in the last step the convergence of the last series can be verified in the same way as (3.3). This proves that

$$\lim_{n \rightarrow \infty} \frac{S_{[\alpha^n]}^{(+)}}{[\alpha^n]} \stackrel{\text{a.s.}}{=} \mu_+. \quad (3.4)$$

For every positive integer  $n$ , there is a unique positive integer  $k_n$  such that  $[\alpha^{k_n-1}] \leq n < [\alpha^{k_n}]$ . Hence,

$$\frac{[\alpha^{k_n-1}]}{[\alpha^{k_n}]} \frac{S_{[\alpha^{k_n-1}]}^{(+)}}{[\alpha^{k_n-1}]} = \frac{S_{[\alpha^{k_n-1}]}^{(+)}}{[\alpha^{k_n}]} \leq \frac{S_n^{(+)}}{n} \leq \frac{S_{[\alpha^{k_n}]}^{(+)}}{[\alpha^{k_n-1}]} = \frac{[\alpha^{k_n}]}{[\alpha^{k_n-1}]} \frac{S_{[\alpha^{k_n}]}^{(+)}}{[\alpha^{k_n}]}. \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$\frac{\mu_+}{\alpha} \stackrel{\text{a.s.}}{\leq} \liminf_{n \rightarrow \infty} \frac{S_n^{(+)}}{n} \leq \limsup_{n \rightarrow \infty} \frac{S_n^{(+)}}{n} \stackrel{\text{a.s.}}{\leq} \alpha \mu_+.$$

By the arbitrariness of  $\alpha$ , we obtain  $S_n^{(+)}/n \xrightarrow{\text{a.s.}} \mu_+$  as  $n \rightarrow \infty$ .

## 4 Applications

### 4.1 Application to Risk Theory

Let  $\{X_k, k = 1, 2, \dots\}$  be a sequence of random variables with partial sums  $S_n, n = 1, 2, \dots$ , and let  $N$  be a nonnegative integer-valued random variable independent of  $\{X_k, k = 1, 2, \dots\}$ .

The study of the tail behavior of the random sum

$$S_N = \sum_{k=1}^N X_k \quad (4.1)$$

is of fundamental interest in various areas of applied probability. Robert and Segers (2008) interpreted  $S_N$  as the total amount of claims of an insurance portfolio in earthquake insurance. Assuming that  $N$  has a consistently-varying tail and that  $X_k$ ,  $k = 1, 2, \dots$ , are independent, identically distributed, and nonnegative, with tails relatively lighter than that of  $N$ , they showed that the tail behavior of  $S_N$  is mainly determined by that of  $N$ . See also Aleškevičienė et al. (2008) and Denisov et al. (2010) for some extensions.

Under the help of Theorem 1.1 and Corollary 2.1, we are able to relax the independence assumption on  $\{X_k, k = 1, 2, \dots\}$  to END. This is particularly relevant in insurance, in view of the fact that claims from an insurance portfolio, or within a given reference period, are generated in the same or similar situations and, hence, they should be dependent.

A distribution  $G$  on  $[0, \infty)$  is said to be of consistent variation, written as  $G \in \mathcal{C}$ , if

$$\lim_{y \rightarrow 1^-} \limsup_{x \rightarrow \infty} \frac{\overline{G}(xy)}{\overline{G}(x)} = 1, \quad \text{or, equivalently,} \quad \lim_{y \rightarrow 1^+} \liminf_{x \rightarrow \infty} \frac{\overline{G}(xy)}{\overline{G}(x)} = 1.$$

Note that the class  $\mathcal{C}$  contains all distributions of regular variation. Clearly, if  $G \in \mathcal{C}$  then, necessarily,  $\overline{G}(xy) = O(\overline{G}(x))$  for every  $y > 0$ . Furthermore, by Lemma 3.5 of Tang and Tsitsiashvili (2003), there is some constant  $p > 0$  such that

$$x^{-p} = o(\overline{G}(x)). \quad (4.2)$$

**Theorem 4.1.** *Consider the random sum (4.1) in which  $\{X_k, k = 1, 2, \dots\}$  is a sequence of END random variables with common distribution  $F$ , mean  $\mu > 0$ , and  $E|X_1|^{1+\delta} < \infty$  for some  $\delta > 0$ , while  $N$  follows a distribution  $G \in \mathcal{C}$ . As  $x \rightarrow \infty$ , the relation*

$$\Pr(S_N > x) \sim \overline{G}(x/\mu) \quad (4.3)$$

holds under one of the following groups of conditions:

- (a)  $x \Pr(|X_1| > x) = o(\overline{G}(x))$ ;
- (b)  $EN < \infty$  and  $\Pr(|X_1| > x) = o(\overline{G}(x))$ .

*Proof.* For arbitrarily fixed  $0 < \varepsilon < 1$ , it is easy to see that

$$\begin{aligned} \Pr(S_N > x) &\leq \sum_{1 \leq n \leq (1-\varepsilon)x/\mu} \Pr(S_n > x) \Pr(N = n) + \Pr(N > (1-\varepsilon)x/\mu) \\ &= I(x) + \Pr(N > (1-\varepsilon)x/\mu). \end{aligned} \quad (4.4)$$

Similarly as in (2.4), for arbitrarily fixed  $v > 0$  define

$$\tilde{X}_k = -vx1_{(X_k - \mu < -vx)} + (X_k - \mu)1_{(-vx \leq X_k - \mu \leq vx)} + vx1_{(X_k - \mu > vx)}, \quad k = 1, 2, \dots$$

Thus,

$$\begin{aligned} I(x) &\leq \sum_{1 \leq n \leq (1-\varepsilon)x/\mu} \Pr(S_n - n\mu > \varepsilon x) \Pr(N = n) \\ &\leq \sum_{1 \leq n \leq (1-\varepsilon)x/\mu} \left( n \Pr(|X_1 - \mu| > vx) + \Pr\left(\sum_{k=1}^n \tilde{X}_k > \varepsilon x\right) \right) \Pr(N = n). \end{aligned} \quad (4.5)$$

For the second term above, by Corollary 2.1(a), for  $v, \varepsilon$ , and  $\delta$  given above and for arbitrarily fixed  $0 < \theta < 1$ , there is some  $K = K(v, \varepsilon, \delta, \theta) > 0$  such that, for all  $n = 1, 2, \dots$ ,

$$\Pr\left(\sum_{k=1}^n \tilde{X}_k > \varepsilon x\right) \leq Kx^{-\delta(1-\theta)/v}. \quad (4.6)$$

Recall relation (4.2). We may suitably adjust the values of  $v$  and  $\theta$  such that  $\delta(1-\theta)/v > p$ . Substituting (4.6) into (4.5), under either condition (a) or condition (b) we have

$$I(x) \leq \Pr(|X_1 - \mu| > vx) \mathbb{E}N1_{(N \leq (1-\varepsilon)x/\mu)} + Kx^{-p} = o(\bar{G}(x)).$$

It follows from this and (4.4) that

$$\limsup_{x \rightarrow \infty} \frac{\Pr(S_N > x)}{\Pr(N > x/\mu)} \leq \lim_{\varepsilon \rightarrow 0^+} \limsup_{x \rightarrow \infty} \frac{\Pr(N > (1-\varepsilon)x/\mu)}{\Pr(N > x/\mu)} = 1. \quad (4.7)$$

To establish the asymptotic lower bound for  $\Pr(S_N > x)$ , with arbitrarily fixed  $0 < \varepsilon < 1$  we derive

$$\begin{aligned} \Pr(S_N > x) &\geq \sum_{n > (1+\varepsilon)x/\mu} \Pr(S_n > x) \Pr(N = n) \\ &\geq \sum_{n > (1+\varepsilon)x/\mu} \Pr\left(\frac{1}{n}S_n - \mu > -\frac{\varepsilon\mu}{1+\varepsilon}\right) \Pr(N = n) \\ &\sim \Pr(N > (1+\varepsilon)x/\mu), \end{aligned}$$

where in the last step we used the fact that, by Theorem 1.1,

$$\lim_{x \rightarrow \infty} \sup_{n > (1+\varepsilon)x/\mu} \left| \Pr\left(\frac{1}{n}S_n - \mu > -\frac{\varepsilon\mu}{1+\varepsilon}\right) - 1 \right|.$$

It follows that

$$\liminf_{x \rightarrow \infty} \frac{\Pr(S_N > x)}{\Pr(N > x/\mu)} \geq \lim_{\varepsilon \rightarrow 0^+} \liminf_{x \rightarrow \infty} \frac{\Pr(N > (1+\varepsilon)x/\mu)}{\Pr(N > x/\mu)} = 1. \quad (4.8)$$

A combination of (4.7) and (4.8) gives the desired asymptotic relation (4.3).  $\square$

## 4.2 Application to Renewal Theory

Let  $\{N_t, t \geq 0\}$  be a quasi-renewal counting process defined as

$$N_t = \max \left\{ n = 1, 2, \dots : \sum_{k=1}^n Y_k \leq t \right\}, \quad t \geq 0, \quad (4.9)$$

where the inter-arrival times  $Y_k, k = 1, 2, \dots$ , form a sequence of nonnegative, END, and identically distributed random variables with common distribution  $G$  and finite, positive mean  $1/\lambda$ .

**Theorem 4.2.** *Consider the quasi-renewal counting process defined by (4.9). As  $t \rightarrow \infty$ ,*

(a)  $N_t/(\lambda t) \xrightarrow{\text{a.s.}} 1;$

(b)  $\text{EN}_t^p \sim (\lambda t)^p$  for every  $p > 0$ .

*Proof.* (a) We follow the proof of Proposition 5.1.4 of Asmussen (2003). By Theorem 1.1(b),

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n Y_k \stackrel{\text{a.s.}}{=} \frac{1}{\lambda}.$$

It trivially follows that  $N_t \xrightarrow{\text{a.s.}} \infty$  as  $t \rightarrow \infty$ . Note that

$$\sum_{k=1}^{N_t} Y_k \leq t < \sum_{k=1}^{N_t+1} Y_k.$$

Dividing each side of the above by  $N_t$  then letting  $t \rightarrow \infty$ , we obtain  $t/N_t \xrightarrow{\text{a.s.}} 1/\lambda$ .

(b) Following the proof of Theorem 1 of Kočetova et al. (2009) with some modifications in relation to the idea used in deriving (2.6), we have, for every  $a > 1$  and some  $b > 1$ ,

$$\lim_{t \rightarrow \infty} \sum_{n > a\lambda t} b^n \Pr(N_t \geq n) = 0. \quad (4.10)$$

Split the moment  $\text{EN}_t^p$  into two parts as

$$\text{EN}_t^p = \left( \sum_{1 \leq n \leq a\lambda t} + \sum_{n > a\lambda t} \right) n^p \Pr(N_t = n) = I_1(t) + I_2(t). \quad (4.11)$$

By the dominated convergence theorem,

$$\lim_{t \rightarrow \infty} \frac{I_1(t)}{(\lambda t)^p} = \lim_{t \rightarrow \infty} \text{E} \left( \frac{N_t}{\lambda t} \right)^p \mathbf{1}_{(1 \leq N_t \leq a\lambda t)} = 1.$$

By (4.10), as  $t \rightarrow \infty$ ,

$$I_2(t) = o(1) \sum_{n > a\lambda t} b^n \Pr(N_t \geq n) = o(1).$$

Substituting these estimates into (4.11) leads to  $\text{EN}_t^p \sim (\lambda t)^p$  as  $t \rightarrow \infty$ .  $\square$

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