Asymptotics for the Ruin Probabilities of a Two-dimensional Renewal Risk Model with Heavy-tailed Claims

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Abstract

In this paper, we consider two dependent classes of insurance business with heavy-tailed claims. The dependence comes from the assumption that claim arrivals of the two classes are governed by a common renewal counting process. We study two types of ruin in the two-dimensional framework. For each type of ruin, we establish an asymptotic formula for the finite-time ruin probability. These formulae possess a certain uniformity feature in the time horizon.

Keywords: Asymptotics; Heavy tail; Finite-time ruin probability; Renewal process; Two-dimensional risk model; Uniformity.

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1 Introduction

In the renewal risk model, the claim amounts $\{X_1, X_2, \ldots\}$ form a sequence of independent, identically distributed (i.i.d.), and non-negative random variables with common distribution F and finite mean. They arrive according to an ordinary renewal counting process $\{N(t), t \geq 0\}$, which has a mean function $\lambda(t) = \mathrm{E}N(t)$ for $t \geq 0$. In other words, $N(t)$ represents the number of claims up to time t. Denote by $\{\theta_1, \theta_2, \ldots\}$ the i.i.d. claim inter-arrival times with common finite positive mean $1/\lambda$. It is assumed that the claim-amount random variables $\{X_1, X_2, \ldots\}$ and the claim-number process $\{N(t), t \geq 0\}$ are independent. With the initial surplus $x \geq 0$ and the constant premium rate $c > 0$, the surplus process is given by

$$R(t) = x + ct - \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{1.1}$$

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where (and throughout) a sum over an empty set of indices equals 0 by convention. Let

$$T(x) = \inf\{t \geq 0 : R(t) < 0 \mid R(0) = x\}$$

be the time of ruin with $T(x) = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. Then, the probability of ruin by time t is defined as a bivariate function

$$\psi(x, t) = \mathrm{P}(T(x) \leq t), \tag{1.2}$$

and the infinite-time ruin probability is defined as

$$\psi(x, \infty) = \lim_{t\to\infty} \psi(x, t) = \mathrm{P}(T(x) < \infty). \tag{1.3}$$

As usual, the safety loading condition takes the form $\mu = c/\lambda - \mathrm{E}X_1 > 0$. Since the renewal risk model is more flexible than the well-known compound Poisson risk model, much research on (1.1) has been carried out in recent years.

In the case that the integrated tail distribution of F is subexponential, Embrechts and Veraverbeke [1] proved that the relation

$$\psi(x, \infty) \sim \frac{1}{\mu} \int_x^\infty \overline{F}(y)\mathrm{d}y \tag{1.4}$$

holds as $x \to \infty$, where $\overline{F} = 1 - F$ and the symbol $\sim$ means that the quotient of both sides tends to 1. See also Veraverbeke [2], Embrechts *et al.* [3], and Zachary [4]. Since this celebrated formula (1.4) for the infinite-time ruin probability (1.3), there has been great interest in the study of asymptotic behavior of the ruin probabilities for heavy-tailed claims.

Let F have a consistently-varying tail, written as $F \in \mathcal{C}$, in the sense that

$$\lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1. \tag{1.5}$$

This class has recently been applied to the study of precise large deviations by several authors; see, for example, Ng *et al.* [5], Tang [6, 7], Wang and Wang [8], Robert and Segers [9], Aleškevičienė *et al.* [10], and references therein. In the literature, some people use the notation $\mathcal{IRV}$ (intermediate regular variation) instead of $\mathcal{C}$ to denote this distribution class. The class $\mathcal{C}$ contains the class $\mathcal{R}$ of distributions with regular variation characterized by the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \tag{1.6}$$

for some $\alpha \geq 0$ and all $y > 1$. In this case, we write $F \in \mathcal{R}_{-\alpha}$. Also, if $F \in \mathcal{C}$, then for any real number l,

$$\overline{F}(x + l) \sim \overline{F}(x), \qquad x \to \infty. \tag{1.7}$$

In fact, relation (1.7) characterizes the class $\mathcal{L}$ of long-tailed distributions. Furthermore, for every $F \in \mathcal{L}$, it is clear that relation (1.7) holds uniformly for all l in a bounded interval. Following Tang and Tsitsiashvili [11], we define

$$\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \qquad J_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y},$$

and call J_F^+ the upper Matuszewska index of the distribution F. It is obvious that $J_F^+ < \infty$ for $F \in \mathcal{C}$ and $J_F^+ = \alpha$ for $F \in \mathcal{R}_{-\alpha}$. Suppose that $F \in \mathcal{C}$ and $\mathrm{E}\theta_1^p < \infty$ for some $p > J_F^+ + 1$. Tang [6] proved that the relation

$$\psi(x,t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \overline{F}(y)\mathrm{d}y \tag{1.8}$$

holds uniformly for all $t \in \Lambda = \{t : \lambda(t) > 0\}$ as $x \to \infty$, that is,

$$\lim_{x\to\infty} \sup_{t\in\Lambda} \left| \frac{\psi(x,t)}{\frac{1}{\mu}\int_x^{x+\mu\lambda(t)} \overline{F}(y)\mathrm{d}y} - 1 \right| = 0.$$

Tang [6] also provided a rather complete list of references on the study of the finite-time ruin probability. See Leipus and Šiaulys [12, 13] for further extensions of this result.

In the past decade, the analysis of dependent classes of insurance business has attracted a vast amount of attention due to their practical importance. Among a lot of works on this topic, we just mention a few that are closely related to the present paper. The ruin problem in high dimensions was first introduced by Collamore [14], who assumed light-tailed claims but general ruin sets. Subsequent results of Collamore [15] lead to estimates of the finite-time ruin probability under the assumption of light-tailed claims. Ruin for multi-dimensional heavy-tailed processes was initially studied by Hult *et al.* [16], who mainly focused on multivariate regularly varying random walks and provided sharp asymptotics for general ruin boundaries. Yuen *et al.* [17] considered two dependent classes of insurance business using the so-called two-dimensional compound Poisson risk model, and discussed various methods for evaluation of ruin probabilities. Li *et al.* [18] extended the two-dimensional compound Poisson risk model by adding a diffusion component. They obtained a Lundberg-type upper bound for the ultimate ruin probability in the case of light-tailed claims, and derived an explicit asymptotic estimate of the finite-time ruin probability in the case of heavy-tailed claims.

In this paper, we aim at extending (1.8) to a two-dimensional renewal risk model for claims with consistently-varying tails. Although the consistent class $\mathcal{C}$ is slightly larger than the commonly-used class $\mathcal{R}$, we expect that the main results of the paper provides insights into the study in the subexponential case. We remark that further extensions to higher-dimensional risk models can be done without much extra work.

The rest of this paper consists of three sections. In Section 2, the model of study is introduced and two asymptotic results are presented. Section 3 states some lemmas that are crucial for the development of the main results. Finally, formal proofs of the main results are given in Section 4.

2 The model and main results

Consider a book of two dependent classes of business. The dependence between the two classes is due to the assumption that they share the same claim-number process. Similar to model (1.1), we take $\{N(t), t \geq 0\}$ to be a common renewal counting process for both classes. The associated i.i.d. claim inter-arrival times are denoted by $\{\theta_1, \theta_2, \ldots\}$. Let $\vec{x} = (x_1, x_2)^\intercal$ be the initial surplus vector and $\vec{c} = (c_1, c_2)^\intercal$ be the vector of premium rate. Let X_1 and X_2 be two independent and non-negative random variables with distributions F_1 and F_2, respectively. In this two-dimensional setting, the surplus process $\vec{R}(t) = (R_1(t), R_2(t))^\intercal$ has the form

$$\vec{R}(t) = \vec{x} + t\vec{c} - \sum_{i=1}^{N(t)} \vec{X}_i, \qquad t \geq 0, \tag{2.1}$$

where the claim-amount vectors $\vec{X}_i = (X_{1i}, X_{2i})^\intercal$, $i = 1, 2, \ldots$, are i.i.d. copies of $\vec{X} = (X_1, X_2)^\intercal$ and independent of $\{N(t), t \geq 0\}$.

As was remarked in Li *et al.* [18], the two-dimensional risk model (2.1) with common arrival process describes the situation that each claim event produces two types of claims. A typical example in motor insurance is that a car accident could cause claims for both the vehicle damage and the injuries of the driver and passengers. This explains the relevance of the two-dimensional risk model in insurance.

In queueing theory, Foss and Korshunov [19] analyzed the asymptotic tail behavior of the stationary waiting time in the GI/GI/2 queue under subexponentiality hypotheses of the service time distribution, and studied the tail asymptotics of a stationary two-dimensional workload vector in the maximal and minimal stability cases. Their queueing model seems analogous to our risk model. Actually, the two models are essentially different and cannot be easily unified. In their queueing model, it is assumed that clients arrive according to a renewal counting process to a system of two or more servers and are to be served in the order of arrivals. In our risk model, it is assumed that each claim event produces two types of claims occurring in two different classes of business. Although both their work and the present paper consider asymptotic tail probabilities of certain quantities under the assumptions of heavy-tailed claims, there is no overlap between the two works.

As is in the one-dimensional case, we assume that θ_i, $i = 1, 2, \ldots$, have a common finite positive mean $1/\lambda$, so that the mean function $\lambda(t) = \mathrm{E}N(t)$ is finite for all $t \geq 0$ but $\lambda(t) \to \infty$ as $t \to \infty$. From the renewal theory (see Proposition 1.4, Chapter V of Asmussen [20]), we have

$$\lambda(t) \sim \lambda t \qquad \text{and} \qquad \frac{N(t)}{\lambda t} \to^p 1, \qquad t \to \infty. \tag{2.2}$$

Furthermore, by the well-known approximation $\mathrm{Var}N(t) = O(t)$ (see Proposition 6.3, Chapter V of Asmussen [20]), it holds that

$$\mathrm{E}N^2(t) = \mathrm{Var}N(t) + [\mathrm{E}N(t)]^2 \sim (\lambda t)^2. \tag{2.3}$$

To avoid the certainty of ruin in each class, we assume that the following safety loading conditions hold

$$\mu_j = \frac{c_j}{\lambda} - \mathrm{E}X_j > 0, \qquad j = 1, 2. \tag{2.4}$$

Define the times of ruin of the two marginal processes as

$$T_j(x_j) = \inf\{t : R_j(t) < 0 \mid R_j(0) = x_j\}, \qquad j = 1, 2.$$

For the two-dimensional model, two kinds of ruin time are investigated. The first one is

$$T_a(\vec{x}) = T_1(x_1) \vee T_2(x_2),$$

representing the first time when both $R_1(t)$ and $R_2(t)$ have ever become negative, while the second one is

$$T_b(\vec{x}) = T_1(x_1) \wedge T_2(x_2),$$

representing the first time when either $R_1(t)$ or $R_2(t)$ becomes negative. Similar to (1.2), we define

$$\psi_a(\vec{x}; t) = \mathrm{P}\left(T_a(\vec{x}) \leq t\right) \qquad \text{and} \qquad \psi_b(\vec{x}; t) = \mathrm{P}\left(T_b(\vec{x}) \leq t\right).$$

We remark that Li *et al.* [18] considered the following two definitions of ruin time:

$$\begin{aligned} T_{\min}(\vec{x}) &= \inf\{t : \min\{R_1(t), R_2(t)\} < 0 \mid \vec{R}(0) = \vec{x}\}, \\ T_{\max}(\vec{x}) &= \inf\{t : \max\{R_1(t), R_2(t)\} < 0 \mid \vec{R}(0) = \vec{x}\}. \end{aligned}$$

Clearly, $T_{\min}(\vec{x})$ and $T_b(\vec{x})$ are the same. However, by comparing $T_a(\vec{x})$ and $T_{\max}(\vec{x})$, we see that $T_{\max}(\vec{x})$ describes a more critical time of the business for which a uniform asymptotic formula for its associated ruin probability is very difficult to derive. Instead, we investigate the ruin probability corresponding to $T_a(\vec{x})$, that is, $\psi_a(\vec{x}; t)$.

In the sequel, unless otherwise stated, the limit procedure is to let $(x_1, x_2) \to (\infty, \infty)$ but with both $x_1 = O(x_2)$ and $x_2 = O(x_1)$. With the above set-up, we are now ready to state our main results which give uniform asymptotic estimates of the finite-time ruin probabilities $\psi_a(\vec{x}; t)$ and $\psi_b(\vec{x}; t)$.

Theorem 2.1. *For the two-dimensional risk model with the safety loading condition (2.4), suppose that both F_1 and F_2 belong to the class $\mathcal{C}$ and that $\mathrm{E}\theta_1^p < \infty$ for some $p > J_{F_1}^+ + J_{F_2}^+ + 1$. Then, it holds uniformly for all $t \in \Lambda$ that*

$$\psi_a(\vec{x}; t) \sim \mathrm{E}\left[\prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j + \mu_j N(t)} \overline{F_j}(y)\mathrm{d}y\right], \tag{2.5}$$

and that

$$\psi_b(\vec{x}; t) \sim \sum_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j + \mu_j \lambda(t)} \overline{F_j}(y)\mathrm{d}y. \tag{2.6}$$

Restricting our attention to a smaller t-region, we can obtain more explicit asymptotic formulae for the two ruin probabilities. We say that $a(x; t) \sim b(x; t)$ holds uniformly for $t \gg 0$ if the relation

$$\lim_{x \to \infty} \sup_{t \in \Delta} \left| \frac{a(x; t)}{b(x; t)} - 1 \right| = 0$$

holds uniformly for all $t \in \Delta = (f(x), \infty)$ for some function $f : [0, \infty) \to [0, \infty)$ with $f(x) \to \infty$ as $x \to \infty$. We end the section by presenting the last main result of the paper.

Theorem 2.2. *Under the conditions of Theorem 2.1, it holds uniformly for $t \gg 0$ that*

$$\psi_a(\vec{x}; t) \sim \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y, \tag{2.7}$$

and that

$$\psi_b(\vec{x}; t) \sim \sum_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y. \tag{2.8}$$

3 Lemmas

This section presents several lemmas that will be used in the next section. Lemma 3.1 comes from Korshunov [21] while Lemmas 3.2 and 3.3 are extracted from Tang [6], who attributed the latter two results to Proposition 2.2.1 of Bingham *et al.* [22] and Theorems 5 and 6 of Kiefer and Wolfowitz [23], respectively.

Lemma 3.1. *Consider a random walk $\{S_n, n = 0, 1, \ldots\}$ whose i.i.d. increments have a common distribution $F \in \mathcal{C}$ and a finite mean $-\nu < 0$. It holds uniformly for $n = 1, 2, \ldots$ that*

$$\mathrm{P}\left(\max_{0\le k\le n} S_k > x\right) \sim \frac{1}{\nu} \int_{x}^{x+n\nu} \overline{F}(y)\mathrm{d}y, \qquad x \to \infty. \tag{3.1}$$

Lemma 3.2. *If $F \in \mathcal{C}$, then $x^{-p} = o\left(\overline{F}(x)\right)$ holds for all $p > J_F^+$.*

Lemma 3.3. *Let $\{X_1, X_2, \ldots\}$ be a sequence of i.i.d. random variables with common negative finite mean and let $M = \max_{0\le k<\infty} \sum_{i=1}^{k} X_i$. Then, for any $p > 1$, the assertions $\mathrm{E}M^{p-1} < \infty$ and $\mathrm{E}\left(\max\{X_1, 0\}\right)^p < \infty$ are equivalent.*

Let $a \le b \le c$ be some constants and let $f(y)$ be a non-increasing function on $[a, c]$. Although the inequality

$$\int_{a}^{c} f(y)\mathrm{d}y \le \frac{c-a}{b-a} \int_{a}^{b} f(y)\mathrm{d}y \tag{3.2}$$

is elementary, we use it several times in the proofs of the following lemma and Theorems 2.1 and 2.2.

Lemma 3.4. *Let $\{N(t), t \ge 0\}$ be a renewal counting process with mean function $\lambda(t)$, $t \ge 0$, and let F belong to the class $\mathcal{L}$ with finite mean. Then, for any $c > 0$, it holds uniformly for all $t \in \Lambda$ that*

$$\mathrm{E}\int_{x}^{x+cN(t)} \overline{F}(y)\mathrm{d}y \sim \int_{x}^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y, \qquad x \to \infty. \tag{3.3}$$

Proof. First of all, we arbitrarily choose a constant $T \in \Lambda$, and prove that relation (3.3) holds uniformly for all $t \in \Lambda_T = [0, T] \cap \Lambda$. On one hand,

$$\mathrm{E}\int_{x}^{x+cN(t)} \overline{F}(y)\mathrm{d}y \le \overline{F}(x)c\lambda(t) \sim \int_{x}^{x+c\lambda(t)} \overline{F}(y)dy, \tag{3.4}$$

uniformly for all $t \in \Lambda_T$ because of the local uniformity of the asymptotics in (1.7). On the other hand, for an arbitrarily chosen integer $N = 1, 2, \ldots$,

$$\begin{aligned} \mathrm{E}\int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y &\geq \sum_{n=1}^{N} \int_x^{x+cn} \overline{F}(y)\mathrm{d}y \mathrm{P}\left(N(t) = n\right) \\ &\geq \overline{F}(x+cN)c\sum_{n=1}^{N} n\mathrm{P}\left(N(t)=n\right) \\ &\sim \overline{F}(x)c\lambda(t)\left(1 - \frac{1}{\lambda(t)}\mathrm{E}N(t)1_{(N(t)\geq N+1)}\right), \end{aligned}$$

uniformly for all $t \in \Lambda_T$ where the last step is again due to (1.7). Therefore,

$$\liminf_{x\to\infty} \inf_{t\in\Lambda_T} \frac{\mathrm{E}\int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y}{\overline{F}(x)c\lambda(t)} \geq 1 - \sup_{t\in\Lambda_T} \frac{1}{\lambda(t)}\mathrm{E}N(t)1_{(N(t)\geq N+1)}. \tag{3.5}$$

Following the proof of Lemma 5.3 of Tang [24], one can show that

$$\lim_{N\to\infty} \sup_{t\in\Lambda_T} \frac{1}{\lambda(t)}\mathrm{E}N(t)1_{(N(t)\geq N+1)} = 0.$$

Since the left-hand side of (3.5) does not involve N, letting $N \to \infty$ on the right-hand side of (3.5) leads to

$$\liminf_{x\to\infty} \inf_{t\in\Lambda_T} \frac{\mathrm{E}\int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y}{\overline{F}(x)c\lambda(t)} \geq 1.$$

Thus, it holds uniformly for all $t \in \Lambda_T$ that

$$\mathrm{E}\int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y \geq (1+o(1))\overline{F}(x)c\lambda(t) \sim \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y. \tag{3.6}$$

It follows from (3.4) and (3.6) that (3.3) holds uniformly for all $t \in \Lambda_T$. In particular, it implies that for any $0 < \varepsilon < 1$, all $t \in \Lambda_T$, and all large x,

$$(1-\varepsilon)\int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y \leq \mathrm{E}\int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y \leq (1+\varepsilon)\int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y. \tag{3.7}$$

We now extend the uniformity of (3.3) to all $t \in \Lambda$. Applying the techniques of Klüppelberg and Mikosch [25], we first derive an asymptotic upper bound for the expectation on the left-hand side of (3.3). For $\varepsilon > 0$, applying (2.2) and the dominated convergence theorem, we have

$$\lim_{t\to\infty} \sum_{n>(1+\varepsilon)\lambda(t)} \frac{n}{\lambda(t)}\mathrm{P}\left(N(t)=n\right) = 1 - \lim_{t\to\infty} \mathrm{E}\frac{N(t)}{\lambda(t)}1_{(N(t)\leq(1+\varepsilon)\lambda(t))} = 0.$$

Therefore, there is some $T \in \Lambda$ such that for all $t \geq T$, the relations

$$\mathrm{P}\left(1-\varepsilon \leq \frac{N(t)}{\lambda(t)} \leq 1+\varepsilon\right) \geq 1-\varepsilon \tag{3.8}$$

and

$$\sum_{n>(1+\varepsilon)\lambda(t)} \frac{n}{\lambda(t)} \mathrm{P}\left(N(t)=n\right) \leq \varepsilon \tag{3.9}$$

hold simultaneously. For any $t \geq T$, we rewrite the left-hand side of (3.3) into two parts as

$$\left(\sum_{1\leq n\leq(1+\varepsilon)\lambda(t)} + \sum_{n>(1+\varepsilon)\lambda(t)}\right) \int_x^{x+cn} \overline{F}(y)\mathrm{d}y \mathrm{P}\left(N(t)=n\right) = I_1(x,t) + I_2(x,t).$$

Because of (3.2), it holds uniformly for all $t \geq T$ that

$$I_1(x,t) \leq \int_x^{x+c(1+\varepsilon)\lambda(t)} \overline{F}(y)\mathrm{d}y \leq (1+\varepsilon) \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y. \tag{3.10}$$

Furthermore, using (3.2) and (3.9), we obtain

$$I_2(x,t) \leq \sum_{n>(1+\varepsilon)\lambda(t)} \frac{n}{\lambda(t)} \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y \mathrm{P}(N(t)=n) \leq \varepsilon \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y,$$

uniformly for all $t \geq T$. These prove that

$$\mathrm{E} \int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y \leq (1+2\varepsilon) \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y, \tag{3.11}$$

uniformly for all $t \geq T$. To complete the proof, we need to derive the corresponding asymptotic lower bound. Similar to the derivation of (3.10), we have

$$\begin{aligned}
\mathrm{E} \int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y &\geq \sum_{(1-\varepsilon)\lambda(t)\leq n\leq(1+\varepsilon)\lambda(t)} \int_x^{x+cn} \overline{F}(y)\mathrm{d}y \mathrm{P}\left(N(t)=n\right) \\
&\geq (1-\varepsilon) \int_x^{x+c(1-\varepsilon)\lambda(t)} \overline{F}(y)\mathrm{d}y \\
&\geq (1-\varepsilon) \left(\int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y - \int_x^{x+c\varepsilon\lambda(t)} \overline{F}(y)\mathrm{d}y\right) \\
&\geq (1-\varepsilon)^2 \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y, \qquad (3.12)
\end{aligned}$$

uniformly for all $t \geq T$ where (3.8) is used in the second step, and (3.2) is used in the fourth step. The conjunction of (3.11) and (3.12) implies that the inequalities

$$(1-\varepsilon)^2 \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y \leq \mathrm{E} \int_x^{x+cN(t)} \overline{F}(y)\mathrm{d}y \leq (1+\varepsilon)^2 \int_x^{x+c\lambda(t)} \overline{F}(y)\mathrm{d}y \tag{3.13}$$

hold for all $x > 0$ and all $t \geq T$. By (3.7) and (3.13), we prove the uniformity of (3.3) over $t \in \Lambda$. □

4 Proofs of the Main Results

In this section, we present the proofs of relations (2.5)-(2.8).

4.1 Proof of Relation (2.5)

Clearly,

$$\begin{aligned}
&\psi_a(\vec{x};t)\\
&= \mathrm{P}\left(\sup_{0<s\le t}\left(\sum_{i=1}^{N(s)} X_{1i}-c_1 s\right)>x_1,\ \sup_{0<s\le t}\left(\sum_{i=1}^{N(s)} X_{2i}-c_2 s\right)>x_2\right)\\
&= \mathrm{P}\left(\sup_{1\le k\le N(t)}\left(\sum_{i=1}^{k}(X_{1i}-c_1\theta_i)\right)>x_1,\ \sup_{1\le k\le N(t)}\left(\sum_{i=1}^{k}(X_{2i}-c_2\theta_i)\right)>x_2\right)\\
&= \sum_{n=1}^{\infty}\mathrm{P}\Bigg(\sup_{1\le k\le n}\left(\sum_{i=1}^{k}(X_{1i}-c_1\theta_i)\right)>x_1,\ \sup_{1\le k\le n}\left(\sum_{i=1}^{k}(X_{2i}-c_2\theta_i)\right)>x_2, N(t)=n\Bigg). \qquad (4.1)
\end{aligned}$$

We first derive an upper bound for $\psi_a(\vec{x};t)$. Choosing some $\varepsilon>0$, we rewrite the safety loading conditions (2.4) as

$$\mu_j^{(-\varepsilon)}=\frac{c_j(1-\varepsilon)}{\lambda}-\mathrm{E}X_j>0, \qquad j=1,2.$$

For arbitrarily chosen $\delta\in(0,1)$, we have

$$\begin{aligned}
&\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1\theta_i)>x_1,\ \sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2\theta_i)>x_2, N(t)=n\right)\\
&\subset\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1(1-\varepsilon)\mathrm{E}\theta_i)+c_1\sup_{1\le k\le n}\sum_{i=1}^{k}((1-\varepsilon)\mathrm{E}\theta_i-\theta_i)>x_1\right)\\
&\quad\bigcap\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2(1-\varepsilon)\mathrm{E}\theta_i)+c_2\sup_{1\le k\le n}\sum_{i=1}^{k}((1-\varepsilon)\mathrm{E}\theta_i-\theta_i)>x_2\right)\\
&\quad\bigcap(N(t)=n)\\
&\subset\Bigg[\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1(1-\varepsilon)\mathrm{E}\theta_i)>(1-\delta)x_1\right)\\
&\quad\bigcap\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2(1-\varepsilon)\mathrm{E}\theta_i)>(1-\delta)x_2\right)\bigcap(N(t)=n)\Bigg]\\
&\quad\bigcup\left(\sup_{1\le k<\infty}\sum_{i=1}^{k}((1-\varepsilon)\mathrm{E}\theta_i-\theta_i)>\delta\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right), N(t)=n\right).
\end{aligned}$$

Since $\{X_{11},X_{12},\ldots\}$, $\{X_{21},X_{22},\ldots\}$, and $\{N(t),t\ge 0\}$ are independent, it follows that

$$\psi_a(\vec{x};t)\le\sum_{n=1}^{\infty}J_1(x_1,n)J_2(x_2,n)\mathrm{P}\left(N(t)=n\right)+J_3(x_1,x_2), \qquad (4.2)$$

where

$$J_1(x_1,n) = \mathrm{P}\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1(1-\varepsilon)\mathrm{E}\theta_i) > (1-\delta)x_1\right),$$
$$J_2(x_2,n) = \mathrm{P}\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2(1-\varepsilon)\mathrm{E}\theta_i) > (1-\delta)x_2\right),$$
$$J_3(x_1,x_2) = \mathrm{P}\left(\sup_{1\le k<\infty}\sum_{i=1}^{k}((1-\varepsilon)\mathrm{E}\theta_i-\theta_i) > \delta\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)\right).$$

Write $M_\varepsilon = \sup_{0\le k<\infty}\sum_{i=1}^{k}((1-\varepsilon)\mathrm{E}\theta_i-\theta_i)$ which is finite almost surely. Since the i.i.d. random variables $(1-\varepsilon)\mathrm{E}\theta_i-\theta_i$, $i=1,2,\ldots$, are bounded from above, it follows from Lemma 3.3 that M_ε has finite moments of arbitrary positive orders. Thus, it holds for every $p>0$ that

$$J_3(x_1,x_2) \le \left(\delta\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)\right)^{-p+1}\mathrm{E}M_\varepsilon^{p-1}1_{\left(M_\varepsilon>\delta\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)\right)} = o(1)\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)^{-p+1}.$$

Since $x_1=O(x_2)$ and $x_2=O(x_1)$, a direct application of Lemma 3.2 gives

$$\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)^{-p+1} = o(1)\overline{F_1}(x_1)\overline{F_2}(x_2),$$

for $p>J_{F_1}^{+}+J_{F_2}^{+}+1$. Therefore, it holds uniformly for all $n=1,2,\ldots$ that

$$J_3(x_1,x_2) = o(1)J_1(x_1,n)J_2(x_2,n). \tag{4.3}$$

Furthermore, applying Lemma 3.1 to $J_1(x_1,n)$ and $J_2(x_2,n)$ yields

$$\begin{aligned} J_j(x_j,n) &\sim \frac{1}{\mu_j^{(-\varepsilon)}}\int_{(1-\delta)x_j}^{(1-\delta)x_j+n\mu_j^{(-\varepsilon)}}\overline{F_j}(y)\mathrm{d}y \\ &\le \frac{1-\delta}{\mu_j^{(-\varepsilon)}}\int_{x_j}^{x_j+\frac{n\mu_j}{1-\delta}}\overline{F_j}\left((1-\delta)y\right)\mathrm{d}y \\ &\le \frac{1}{\mu_j^{(-\varepsilon)}}\sup_{y>x_j}\frac{\overline{F_j}((1-\delta)y)}{\overline{F_j}(y)}\int_{x_j}^{x_j+n\mu_j}\overline{F_j}(y)\mathrm{d}y, \end{aligned} \tag{4.4}$$

uniformly for all $n=1,2,\ldots$ and $j=1,2$, where relation (3.2) is used in the last step. Then, we substitute (4.3) and (4.4) into (4.2) to obtain

$$\psi_a(\vec{x};t) \le \sum_{n=1}^{\infty}\prod_{j=1}^{2}\frac{1+o(1)}{\mu_j^{(-\varepsilon)}}\sup_{y>x_j}\frac{\overline{F_j}((1-\delta)y)}{\overline{F_j}(y)}\int_{x_j}^{x_j+n\mu_j}\overline{F_j}(y)\mathrm{d}y\mathrm{P}\left(N(t)=n\right).$$

Since ε and δ are arbitrary and $F_1,F_2\in\mathcal{C}$, we have

$$\psi_a(\vec{x};t) \le \frac{1+o(1)}{\mu_1\mu_2}\sum_{n=1}^{\infty}\prod_{j=1}^{2}\int_{x_j}^{x_j+n\mu_j}\overline{F_j}(y)\mathrm{d}y\mathrm{P}\left(N(t)=n\right), \tag{4.5}$$

uniformly for all $t \in \Lambda$.

We now switch our attention to construct the asymptotic lower bound of $\psi_a(\vec{x};t)$. We still start with (4.1). Arbitrarily choose $\varepsilon > 0$ and $\delta \in (0,1)$. Since $\mu_j^{(\varepsilon)} = c_j(1+\varepsilon)/\lambda - \mathrm{E}X_j > 0$ for $j = 1, 2$, we have

$$\begin{aligned}
&\left(\sup_{1\le k\le n}\left(\sum_{i=1}^{k}(X_{1i}-c_1\theta_i)\right) > x_1,\ \sup_{1\le k\le n}\left(\sum_{i=1}^{k}(X_{2i}-c_2\theta_i)\right) > x_2, N(t)=n\right)\\
\supset&\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1(1+\varepsilon)\mathrm{E}\theta_i)+c_1\inf_{1\le k\le n}\sum_{i=1}^{k}((1+\varepsilon)\mathrm{E}\theta_i-\theta_i) > x_1\right)\\
&\bigcap\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2(1+\varepsilon)\mathrm{E}\theta_i)+c_2\inf_{1\le k\le n}\sum_{i=1}^{k}((1+\varepsilon)\mathrm{E}\theta_i-\theta_i) > x_2\right)\\
&\bigcap(N(t)=n)\\
\supset&\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1(1+\varepsilon)\mathrm{E}\theta_i) > (1+\delta)x_1\right)\\
&\bigcap\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2(1+\varepsilon)\mathrm{E}\theta_i) > (1+\delta)x_2\right)\\
&\bigcap\left(\inf_{1\le k<\infty}\sum_{i=1}^{k}((1+\varepsilon)\mathrm{E}\theta_i-\theta_i) \ge -\delta\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)\right)\\
&\bigcap(N(t)=n)\,.
\end{aligned}$$

Analogous to (4.2), it is easy to see that

$$\psi_a(\vec{x};t) \ge \sum_{n=1}^{\infty} J_4(x_1,n)J_5(x_2,n)\mathrm{P}\left(N(t)=n\right) - J_6(x_1,x_2), \tag{4.6}$$

where

$$\begin{aligned}
J_4(x_1,n) &= \mathrm{P}\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{1i}-c_1(1+\varepsilon)\mathrm{E}\theta_i) > (1+\delta)x_1\right),\\
J_5(x_2,n) &= \mathrm{P}\left(\sup_{1\le k\le n}\sum_{i=1}^{k}(X_{2i}-c_2(1+\varepsilon)\mathrm{E}\theta_i) > (1+\delta)x_2\right),\\
J_6(x_1,x_2) &= \mathrm{P}\left(\sup_{1\le k<\infty}\sum_{i=1}^{k}(\theta_i-(1+\varepsilon)\mathrm{E}\theta_i) > \delta\left(\frac{x_1}{c_1}\wedge\frac{x_2}{c_2}\right)\right).
\end{aligned}$$

Parallel to the derivation of the asymptotic upper bound, it follows from (4.6) and the condition $\mathrm{E}\theta_1^p < \infty$ for some $p > J_{F_1}^+ + J_{F_2}^+ + 1$ that

$$\psi_a(\vec{x};t) \ge \frac{1+o(1)}{\mu_1\mu_2}\sum_{n=1}^{\infty}\prod_{j=1}^{2}\int_{x_j}^{x_j+n\mu_j}\overline{F_j}(y)\mathrm{d}y\mathrm{P}\left(N(t)=n\right), \tag{4.7}$$

uniformly for all $t \in \Lambda$. Thus, by (4.5) and (4.7), we conclude that (2.5) holds uniformly for all $t \in \Lambda$.

4.2 Proof of Relation (2.6)

By definition,

$$\psi_b(\vec{x};t) = \mathrm{P}\left(T_b(\vec{x}) \leq t\right) = \mathrm{P}\left(T_1(x_1) \leq t\right) + \mathrm{P}\left(T_2(x_2) \leq t\right) - \mathrm{P}\left(T_a(\vec{x}) \leq t\right).$$

From (1.8), we see that

$$\mathrm{P}\left(T_j(x_j) \leq t\right) \sim \frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda(t)} \overline{F_j}(y)\mathrm{d}y, \qquad j = 1, 2,$$

uniformly for all $t \in \Lambda$. Hence, it suffices to prove that, uniformly for all $t \in \Lambda$,

$$\mathrm{P}\left(T_a(\vec{x}) \leq t\right) = o(1)\sum_{j=1}^{2}\frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda(t)} \overline{F_j}(y)\mathrm{d}y. \tag{4.8}$$

Making use of (2.5), we get

$$\mathrm{P}\left(T_a(\vec{x}) \leq t\right) \leq \frac{1+o(1)}{\mu_1\mu_2}\mathrm{E}\left[\int_{x_1}^{x_1+\mu_1 N(t)} \overline{F_1}(y)\mathrm{d}y\right]\int_{x_2}^{\infty} \overline{F_2}(y)\mathrm{d}y = o(1)\mathrm{E}\int_{x_1}^{x_1+\mu_1 N(t)} \overline{F_1}(y)\mathrm{d}y,$$

uniformly for all $t \in \Lambda$. This together with Lemma 3.4 give relation (4.8).

4.3 Proof of Relation (2.7)

By Theorem 2.1, it suffices to prove that the relation

$$\mathrm{E}\left[\prod_{j=1}^{2}\frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j N(t)} \overline{F_j}(y)\mathrm{d}y\right] \sim \prod_{j=1}^{2}\frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y \tag{4.9}$$

holds uniformly for $t \gg 0$. This can be done by using arguments similar to those in the proof of Lemma 3.4.

For arbitrarily fixed $0 < \varepsilon < 1$, we divide the left-hand side of (4.9) into three parts given by

$$\begin{aligned}&\left(\sum_{1\leq n<(1-\varepsilon)\lambda t} + \sum_{(1-\varepsilon)\lambda t\leq n\leq(1+\varepsilon)\lambda t} + \sum_{n>(1+\varepsilon)\lambda t}\right)\prod_{j=1}^{2}\frac{1}{\mu_j}\int_{x_j}^{x_j+n\mu_j} \overline{F_j}(y)\mathrm{d}y\mathrm{P}\left(N(t)=n\right)\\ &= K_1(x_1,x_2,t) + K_2(x_1,x_2,t) + K_3(x_1,x_2,t).\end{aligned} \tag{4.10}$$

First, we have

$$\begin{aligned}K_1(x_1,x_2,t) &\leq \prod_{j=1}^{2}\frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y\mathrm{P}\left(N(t) < (1-\varepsilon)\lambda t\right)\\ &= o(1)\prod_{j=1}^{2}\frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y.\end{aligned} \tag{4.11}$$

uniformly for $t \gg 0$. Similarly, uniformly for $t \gg 0$,

$$\begin{aligned} K_2(x_1,x_2,t) &\le \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+(1+\varepsilon)\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y \mathrm{P}\left((1-\varepsilon)\lambda t \le N(t) \le (1+\varepsilon)\lambda t\right) \\ &\sim \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+(1+\varepsilon)\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y \\ &\le (1+\varepsilon)^2 \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y. \end{aligned} \tag{4.12}$$

Furthermore, by (2.2), (2.3), and the dominated convergence theorem, we have

$$\lim_{t\to\infty} \sum_{n>(1+\varepsilon)\lambda t} \frac{n^2}{(\lambda t)^2} \mathrm{P}\left(N(t)=n\right) = \lim_{t\to\infty} \left[\mathrm{E}\left(\frac{N(t)}{\lambda t}\right)^2 - \mathrm{E}\left(\frac{N(t)}{\lambda t}\right)^2 1_{(N(t)\le(1+\varepsilon)\lambda t)} \right] = 0,$$

which leads to

$$\begin{aligned} K_3(x_1,x_2,t) &\le \left(\sum_{n>(1+\varepsilon)\lambda t} \frac{n^2}{(\lambda t)^2} \mathrm{P}\left(N(t)=n\right) \right) \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y \\ &= o(1) \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y, \end{aligned} \tag{4.13}$$

uniformly for $t \gg 0$. Hence, (4.10)-(4.13) imply that uniformly for $t \gg 0$,

$$\mathrm{E}\left[\prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j N(t)} \overline{F_j}(y)\mathrm{d}y \right] \le (1+o(1)) \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y. \tag{4.14}$$

It is easy to construct the corresponding asymptotic lower bound. In fact, following the derivation of (4.12), one can show that the inequalities

$$\mathrm{E}\left[\prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j N(t)} \overline{F_j}(y)\mathrm{d}y \right] \ge K_2(x_1,x_2,t) \ge (1+o(1)) \prod_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F_j}(y)\mathrm{d}y \tag{4.15}$$

hold uniformly for $t \gg 0$. Finally, it follows from (4.14) and (4.15) that relation (4.9) holds uniformly for $t \gg 0$.

4.4 Proof of Relation (2.8)

To prove (2.8), it suffices to prove that the following relation

$$\sum_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda(t)} \overline{F}_j(y)\mathrm{d}y \sim \sum_{j=1}^{2} \frac{1}{\mu_j} \int_{x_j}^{x_j+\mu_j\lambda t} \overline{F}_j(y)\mathrm{d}y \tag{4.16}$$

holds uniformly for $t \gg 0$ because of Theorem 2.1. From (2.2), it holds for every $\varepsilon > 0$ and all large t that $(1-\varepsilon)\lambda t \leq \lambda(t) \leq (1+\varepsilon)\lambda t$. Hence, uniformly for $t \gg 0$,

$$\frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda(t)} \overline{F}_j(y)\mathrm{d}y \leq \frac{1}{\mu_j}\int_{x_j}^{x_j+\mu_j(1+\varepsilon)\lambda t} \overline{F}_j(y)\mathrm{d}y \leq \frac{1+\varepsilon}{\mu_j}\int_{x_j}^{x_j+\mu_j\lambda t} \overline{F}_j(y)\mathrm{d}y,$$

for $j = 1, 2$. Likewise, we can establish the corresponding lower bound. Mimicking the proof of (4.9), one can show that (4.16) holds uniformly for $t \gg 0$.

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