# The Finite-time Ruin Probability with Dependent Insurance and Financial Risks

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## Abstract

Consider a discrete-time insurance risk model. Within period $i$, the net insurance loss is denoted by a real-valued random variable $X_i$. The insurer makes both risk-free and risky investments, leading to an overall stochastic discount factor $Y_i$ from time $i$ to time $i-1$. Assume that $(X_i, Y_i)$, $i \in \mathbb{N}$, form a sequence of independent and identically distributed random pairs following a common bivariate Farlie-Gumbel-Morgenstern distribution with marginal distribution functions $F$ and $G$. When $F$ is subexponential and $G$ fulfills some constraints in order for the product convolution of $F$ and $G$ to be subexponential too, we derive a general asymptotic formula for the finite-time ruin probability. Then for special cases in which $F$ belongs to the Fréchet or Weibull max-domain of attraction we improve this general formula to be transparent.

*Keywords*: Asymptotics; Farlie-Gumbel-Morgenstern distribution; Finite-time ruin probability; Maximum-domain of attraction; Subexponential distribution

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## 1 Introduction

Following the works of Nyrhinen (1999, 2001) and Tang and Tsitsiashvili (2003, 2004), we consider a discrete-time insurance risk model. Within period $i$, the net insurance loss (equal to the total claim amount minus the total premium income) is denoted by a real-valued random variable $X_i$, $i \in \mathbb{N}$. Suppose that the insurer makes both risk-free and risky investments, which lead to an overall stochastic discount factor $Y_i$ from time $i$ to time $i-1$. See, e.g. Section 4.1 of Hashorva et al. (2010) for the structure of these overall stochastic discount factors. In the terminology of Tang and Tsitsiashvili (2003, 2004), we call $\{X_i, i \in \mathbb{N}\}$ insurance risks and call $\{Y_i, i \in \mathbb{N}\}$ financial risks. Thus, the sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n \in \mathbb{N}, \tag{1.1}$$

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represents the stochastic present value of aggregate net losses up to time $n$. As usual, the probability of ruin by time $n$ is defined to be

$$\psi(x;n) = \Pr\left(\max_{1\le m\le n}\sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x\right), \qquad n\in\mathbb{N}, \tag{1.2}$$

where $x \ge 0$ is interpreted as the initial wealth of the insurer.

Initiated by the work of Tang and Tsitsiashvili (2003), there has been a vast amount of literature studying the asymptotic behavior of the ruin probability of this risk model in the presence of heavy-tailed insurance or/and financial risks. In the study, it is common to assume that both $\{X_i, i\in\mathbb{N}\}$ and $\{Y_i, i\in\mathbb{N}\}$ are sequences of independent and identically distributed (i.i.d.) random variables and they are independent of each other as well. Undoubtedly, this assumption of complete independence is far unrealistic. A recent new trend of the study is to introduce various dependence structures to the risk model. In this direction, we refer the reader to Goovaerts et al. (2005), Tang (2006a), Zhang et al. (2009), Weng et al. (2009) and Yi et al. (2011), among many others. However, there are few papers which take into account the dependence between insurance and financial risks, with the difficulty existing in describing the tail behavior of the product of dependent random variables.

In the present paper we assume that $(X_i, Y_i)$, $i\in\mathbb{N}$, form a sequence of i.i.d. random pairs with a generic random pair $(X, Y)$ whose components are however dependent. We use a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution to model the dependence structure of $(X, Y)$. Recall that a bivariate FGM distribution function is of the form

$$\Pi\left(x,y\right) = F(x)G(y)\left(1+\theta\overline{F}(x)\overline{G}(y)\right), \tag{1.3}$$

where $F$ and $G$ are marginal distribution functions and $\theta$ is a real number fulfilling $|\theta|\le 1$ in order for $\Pi(\cdot,\cdot)$ to be a proper bivariate distribution function. Trivially, if $\theta = 0$ then (1.3) describes a joint distribution function of two independent random variables. We refer the reader to Kotz et al. (2000) for a general account on FGM distribution functions and to Tang and Vernic (2007) and Cossette et al. (2008) for applications of FGM distribution functions to risk theory.

First, under the assumption that $F$ in (1.3) is a subexponential distribution function while $G$ fulfills some constraints in order for the product convolution of $F$ and $G$ to be a subexponential distribution function too, we derive a general asymptotic formula for the ruin probability $\psi(x;n)$. Here the product convolution of $F$ and $G$, denoted as $F\otimes G$, is understood as the distribution function of the product $X^*Y^*$, where $X^*$ and $Y^*$ are two independent random variables with $X^*$ identically distributed as $X$ and $Y^*$ as $Y$.

Next, for some special cases in which $F$ is a distribution function in the Fréchet or Weibull max-domain of attraction, we improve the general asymptotic formula to be transparent. The obtained asymptotic formulas successfully capture the impact of the underlying dependence structure of $(X, Y)$.

The rest of this paper consists of three sections. Section 2 prepares preliminaries of subexponential distributions and maximum-domains of attraction, Section 3 presents the main results and Section 4 proves these results.

## 2 Preliminaries

Throughout the paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \leq 1$ and write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$. We also write $f(x) \asymp g(x)$ if $0 < \liminf f(x)/g(x) \leq \limsup f(x)/g(x) < \infty$.

A distribution function $F$ on $\mathbb{R}^+ = [0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F}(x) = 1 - F(x) > 0$ for all $x \in \mathbb{R}^+$ and

$$\overline{F^{2*}}(x) \sim 2\overline{F}(x),$$

where $F^{2*}$ denotes the two-fold convolution of $F$. More generally, a distribution function $F$ on $\mathbb{R}$ is still said to be subexponential if the distribution function $F_+(x) = F(x)1_{(x \geq 0)}$ is subexponential. It is well known that every subexponential distribution function $F$ is long tailed, written as $F \in \mathcal{L}$, in the sense that the relation $\overline{F}(x + y) \sim \overline{F}(x)$ holds for some (or, equivalently, for all) $y \neq 0$; see Lemma 1.3.5(a) of Embrechts et al. (1997). Following the works of Konstantinides et al. (2002) and Tang (2006b), we say that a distribution function $F$ on $\mathbb{R}$ belongs to the class $\mathcal{A}$ if $F \in \mathcal{S}$ and the relation

$$\limsup_{x \to \infty} \frac{\overline{F}(cx)}{\overline{F}(x)} < 1 \tag{2.1}$$

holds for some $c > 1$. Note that (2.1) is really a mild constraint and is fulfilled by almost all useful distribution functions with unbounded supports. Thus, as remarked by Tang (2006b), the class $\mathcal{A}$ almost exhausts the class $\mathcal{S}$.

Under the assumption that the insurance risk $X$ follows a distribution function $F \in \mathcal{S}$ or $\mathcal{A}$, we shall derive a general asymptotic formula for the ruin probability $\psi(x; n)$. In order to further improve this general formula to be transparent, we need to introduce several concepts in extreme value theory.

A distribution function $F$ on $\mathbb{R}$ is said to belong to the max-domain of attraction of an extreme value distribution function $F_0$, written as $F \in \mathrm{MDA}(F_0)$, if

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - F_0(x)| = 0 \tag{2.2}$$

holds for some constants $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$. Only three choices for $F_0$ in (2.2) are possible. They are the Fréchet, Gumbel and Weibull distributions, which we denote by $\Phi_\alpha$, $\Lambda$ and $\Psi_\alpha$ for $\alpha > 0$, respectively. A distribution function $F$ belongs to $\mathrm{MDA}(\Phi_\alpha)$ if and only if $\overline{F}(\cdot)$ is regularly varying at infinity with index $-\alpha$, that is, the relation

$$\lim_{x\to\infty}\frac{\overline{F}(cx)}{\overline{F}(x)}=c^{-\alpha}$$

holds for all $c>0$; see Theorem 3.3.7 of Embrechts et al. (1997). Hence, $\mathrm{MDA}(\Phi_\alpha)\subset\mathcal{S}$; see Lemma 1.3.1 of Embrechts et al. (1997). A distribution function $F$ with an upper endpoint $\hat{x}=\sup\{x\in\mathbb{R}:F(x)<1\}\le\infty$ belongs to $\mathrm{MDA}(\Lambda)$ if and only if the relation

$$\lim_{x\uparrow\hat{x}}\frac{\overline{F}(x+ca(x))}{\overline{F}(x)}=\mathrm{e}^{-c}$$

holds for some positive auxiliary function $a(\cdot)$ and all $c\in\mathbb{R}$. Recall that the auxiliary function $a(\cdot)$ is asymptotically equivalent to the mean excess function of $F$, namely $a(x)\sim\mathrm{E}\left[X-x|X>x\right]$ as $x\uparrow\hat{x}$; see Theorem 3.3.27 of Embrechts et al. (1997). Almost all distribution functions with rapidly-varying tails are included by $\mathrm{MDA}(\Lambda)$. A distribution function $F$ belongs to $\mathrm{MDA}(\Psi_\alpha)$ if and only if its upper endpoint $\hat{x}$ is finite and the relation

$$\lim_{x\to\infty}\frac{\overline{F}(\hat{x}-c/x)}{\overline{F}(\hat{x}-1/x)}=c^{\alpha}$$

holds for all $c>0$; see Theorem 3.3.12 of Embrechts et al. (1997).

## 3 Main Results

Recall the insurance risk model introduced in Section 1. In the sequel, denote by $F$ on $\mathbb{R}$, $G$ on $\mathbb{R}^+$ and $H$ on $\mathbb{R}^+$ the distribution functions of $X$, $Y$ and $XY$, respectively. Throughout the rest of the paper, $F$ is assumed to be heavy tailed and, to avoid triviality, $G$ is assumed to be non-degenerate at 0. For each $i\in\mathbb{N}$, denote by $H_i$ the distribution function of $X_i\prod_{j=1}^{i}Y_j$. Note that $H=H_1$.

Here is our first main result:

**Theorem 3.1** *Assume that $(X_i,Y_i)$, $i\in\mathbb{N}$, form a sequence of i.i.d. random pairs following a common bivariate FGM distribution function (1.3) with $\theta\in(-1,1]$. Further assume that $F\in\mathcal{S}$ and that there is a function $b(\cdot):\mathbb{R}^+\mapsto\mathbb{R}^+$ such that the following items hold simultaneously:*

*(b1)* $b(x)\uparrow\infty$,

*(b2)* $b(x)/x\downarrow 0$,

*(b3)* $\overline{G}\left(b(x)\right)=o\left(\overline{H}(x)\right)$, *and*

*(b4)* $\overline{F}\left(x-b(x)\right)\sim\overline{F}(x)$.

*Then for each $n\in\mathbb{N}$, the distribution function of $S_n$ belongs to the class $\mathcal{S}$ and the ruin probability $\psi(x;n)$ satisfies*

$$\psi(x;n)\sim\Pr\left(S_n>x\right)\sim\sum_{i=1}^{n}\overline{H_i}(x). \tag{3.1}$$

The assumptions (b1)–(b4) were first proposed by Cline and Samorodnitsky (1994) when establishing subexponentiality for the product of independent random variables; see Lemma 4.1 below. These assumptions automatically hold if $Y$ is bounded above. Recall that $F \otimes G$ stands for the product convolution of $F$ and $G$. With $\theta \in (-1, 1]$, relations (4.4) and (4.7) below show that $\overline{H}(x) \asymp \overline{F \otimes G}(x)$. Thus, for this case, the set of assumptions (b1)–(b3) is equivalent to the set of assumptions (b1)–(b2) and

(b3*) $\overline{G}\left(b(x)\right) = o\left(\overline{F \otimes G}(x)\right)$.

Of course (b3*) is easier to verify than (b3).

As remarked by Tang (2006b), (b3) requests that $b(\cdot)$ diverge to $\infty$ not too slowly while (b4) requests that it diverge to $\infty$ not too fast. Very often (b4) appears to be too restrictive for applications. In our second main result below we drop this annoying assumption:

**Theorem 3.2** *Assume that $(X_i, Y_i)$, $i \in \mathbb{N}$, form a sequence of i.i.d. random pairs following a common bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Further assume that $F \in \mathcal{A}$ and that there is a function $b(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that (b1)–(b3) hold simultaneously. Then for each $n \in \mathbb{N}$, the distribution function of $S_n$ belongs to the class $\mathcal{A}$ and the ruin probability $\psi(x; n)$ satisfies (3.1).*

Interestingly, as summarized by Lemma 4.6 below, the set of assumptions (b1)–(b3) amounts to the assertion that the relation $\overline{G}(cx) = o\left(\overline{H}(x)\right)$ holds for all $c > 0$. This latter assertion is much easier to verify than (b1)–(b3).

We end this section with a corollary of Theorems 3.1 and 3.2 in which we focus on improving the asymptotic formula given in (3.1) to be completely transparent within max-domains of attraction:

**Corollary 3.1** *Assume that $(X_i, Y_i)$, $i \in \mathbb{N}$, form a sequence of i.i.d. random pairs following a common bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$.*

*(i) If $F \in \mathrm{MDA}(\Phi_\alpha)$ for some $\alpha > 0$ and $\mathrm{E}\left[Y^{\alpha+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$, then*

$$\psi(x; n) \sim \left((1-\theta)\mathrm{E}\left[Y^\alpha\right] + \theta \mathrm{E}\left[(Y_1^* \vee Y_2^*)^\alpha\right]\right) \frac{1 - \left(\mathrm{E}\left[Y^\alpha\right]\right)^n}{1 - \mathrm{E}\left[Y^\alpha\right]} \overline{F}(x), \tag{3.2}$$

*where $Y_1^*$ and $Y_2^*$ are two independent random variables identically distributed as $Y$.*

*(ii) If $F \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ with an auxiliary function $a(\cdot)$ and $G \in \mathrm{MDA}(\Psi_\alpha)$ for some $\alpha > 0$ with an upper endpoint $0 < \hat{y} < \infty$, then*

$$\psi(x; n) \sim (1+\theta) \sum_{i=1}^n \overline{F}\left(\frac{x}{\hat{y}^i}\right) \left(\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^{i+1}}{x} a\left(\frac{x}{\hat{y}^i}\right)\right)\right)^i. \tag{3.3}$$

Relation (3.2) suggests that, if further $\mathrm{E}\left[Y^\alpha\right] < 1$, the following simple asymptotic formula for the infinite-time ruin probability should hold:

$$\psi(x; \infty) = \Pr\left(\max_{1 \le m < \infty} \sum_{i=1}^m X_i \prod_{j=1}^i Y_j > x\right) \sim \frac{(1-\theta)\mathrm{E}\left[Y^\alpha\right] + \theta \mathrm{E}\left[(Y_1^* \vee Y_2^*)^\alpha\right]}{1 - \mathrm{E}\left[Y^\alpha\right]} \overline{F}(x).$$

Indeed, this can be easily proven by Theorem 1 of Grey (1994). We omit details here. Also note that, when $\hat{y} = 1$, relation (3.3) admits the following substantial simplification:

$$\psi(x;n) \sim (1+\theta)\,\Gamma(\alpha+1)\overline{F}\,(x)\,\overline{G}\left(1-\frac{a(x)}{x}\right). \tag{3.4}$$

Since the right-hand side of (3.4) does not involve $n$, the same asymptotic formula should hold for the infinite-time ruin probability, though we still cannot prove it at this stage.

# 4 Proofs

Recall relations (1.1) and (1.2). Clearly,

$$\Pr\left(\sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j > x\right) \le \psi(x;n) \le \Pr\left(\sum_{i=1}^{n} X_i^{+} \prod_{j=1}^{i} Y_j > x\right), \tag{4.1}$$

where $X_i^{+} = X_i 1_{(X_i \ge 0)}$ denotes the positive part of $X_i$, $i \in \mathbb{N}$. For the proof of (3.1) in both Theorems 3.1 and 3.2, if we can establish the second asymptotic formula in (3.1) while doing so does not require to assume $F(0-) > 0$, then the same asymptotic formula should hold for the right-hand side of (4.1) as well. In this way we will have completed the proof of (3.1). Furthermore, we notice the simple fact

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \overset{\mathrm{d}}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j, \qquad n \in \mathbb{N}, \tag{4.2}$$

due to the i.i.d. assumption for the sequence $\{(X_i, Y_i), i \in \mathbb{N}\}$, where $\overset{\mathrm{d}}{=}$ stands for equality in distribution. Denote by $T_n$ the sum on the right-hand side of (4.2). Therefore, we only need to prove the relation

$$\Pr\left(T_n > x\right) \sim \sum_{i=1}^{n} \overline{H_i}(x). \tag{4.3}$$

## 4.1 Proof of Theorem 3.1

The following first lemma is a restatement of Theorem 2.1 of Cline and Samorodnitsky (1994), which is crucial for establishing our Theorem 3.1:

**Lemma 4.1** *Let $X$ and $Y$ be two independent random variables with distribution functions $F$ on $\mathbb{R}$ and $G$ on $\mathbb{R}^+$, respectively. Denote by $H$ the distribution function of their product $XY$. We have $H \in \mathcal{S}$ if $F \in \mathcal{S}$ and there is a function $b(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that (b1)–(b4) hold simultaneously, namely:*

*(b1)* $b(x) \uparrow \infty$,
*(b2)* $b(x)/x \downarrow 0$,
*(b3)* $\overline{G}\,(b(x)) = o\left(\overline{H}(x)\right)$, *and*
*(b4)* $\overline{F}\,(x - b(x)) \sim \overline{F}(x)$.

Recall the generic random pair $(X, Y)$ in the insurance risk model in Section 1. Introduce six independent random variables $X^*$, $X_1^*$, $X_2^*$, $Y^*$, $Y_1^*$ and $Y_2^*$ with the first three identically distributed as $X$ and the last three as $Y$. In the proof of Theorem 3.1 we also need the following lemma:

**Lemma 4.2** *Let $(X, Y)$ follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Under (b1)–(b3), it holds that*

$$\overline{H}(x) \sim (1-\theta) \Pr\left(X^* Y^* > x\right) + \theta \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right). \tag{4.4}$$

**Proof.** We notice the decomposition

$$\Pi = (1+\theta) FG - \theta F^2 G - \theta F G^2 + \theta F^2 G^2. \tag{4.5}$$

Clearly, $X_1^* \vee X_2^*$ is distributed by $F^2$ and $Y_1^* \vee Y_2^*$ by $G^2$. Thus, it follows from (4.5) that

$$\begin{aligned}\overline{H}(x) &= \Pr\left(XY > x\right) \\ &= (1+\theta) \Pr\left(X^* Y^* > x\right) - \theta \Pr\left(\left(X_1^* \vee X_2^*\right) Y^* > x\right) \\ &\quad -\theta \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right) + \theta \Pr\left(\left(X_1^* \vee X_2^*\right)\left(Y_1^* \vee Y_2^*\right) > x\right).\end{aligned} \tag{4.6}$$

In terms of the function $b(\cdot)$ in (b1)–(b3), we have

$$\begin{aligned}\Pr\left(\left(X_1^* \vee X_2^*\right) Y^* > x\right) &= \int_0^{b(x)} \Pr\left(X_1^* \vee X_2^* > \frac{x}{y}\right) \mathrm{d}G(y) + O\left(\overline{G}(b(x))\right) \\ &= (2+o(1)) \int_0^{b(x)} \Pr\left(X^* > \frac{x}{y}\right) \mathrm{d}G(y) + O\left(\overline{G}(b(x))\right) \\ &= (2+o(1)) \Pr\left(X^* Y^* > x\right) + o\left(\overline{H}(x)\right).\end{aligned}$$

Similarly as above,

$$\Pr\left(\left(X_1^* \vee X_2^*\right)\left(Y_1^* \vee Y_2^*\right) > x\right) = (2+o(1)) \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right) + o\left(\overline{H}(x)\right).$$

Substituting these estimates into (4.6) yields that

$$\overline{H}(x) = (1-\theta+o(1)) \Pr\left(X^* Y^* > x\right) + (\theta+o(1)) \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right) + o\left(\overline{H}(x)\right),$$

which implies relation (4.4). ■

When $\theta \in (-1, 1]$, we apply the two-sided inequality

$$\Pr\left(X^* Y^* > x\right) \leq \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right) \leq 2 \Pr\left(X^* Y^* > x\right)$$

to obtain an asymptotic estimate for the right-hand side of (4.4) as

$$(1-\theta) \Pr\left(X^* Y^* > x\right) + \theta \Pr\left(X^* \left(Y_1^* \vee Y_2^*\right) > x\right) \asymp \Pr\left(X^* Y^* > x\right). \tag{4.7}$$

Therefore, there is no problem with the implication in the last step of the proof of Lemma 4.2. When $\theta = -1$, however, one needs to be aware that the right-hand side of (4.4) could reduce to $o\left(\Pr\left(X^*Y^* > x\right)\right)$, e.g. when $F$ is rapidly-varying tailed and $G$ does not assign a mass at the upper endpoint $0 < \hat{y} \leq \infty$. For this case, (4.4) may fail to give an exact asymptotic relationship. This explains why we have to exclude $\theta = -1$ in Lemma 4.2 and hereby in Theorems 3.1 and 3.2.

Lemma 4.2 offers the following insight. For $\theta \in (-1, 1]$, introduce a function $G_\theta(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ as

$$G_\theta(y) = (1 - \theta)\, G(y) + \theta G^2(y), \tag{4.8}$$

which clearly defines a proper distribution function on $\mathbb{R}^+$. Further introduce a nonnegative random variable $Z_\theta$ distributed by $G_\theta$ and independent of $(X, Y)$. Then relation (4.4) can be rewritten as

$$\Pr\left(XY > x\right) \sim \Pr\left(XZ_\theta > x\right). \tag{4.9}$$

In this way, the dependence structure of $(X, Y)$ is dissolved.

By Lemma 4.1, it is easy to establish the subexponentiality of the independent product $XZ_\theta$ in (4.9). Also recall that the class $\mathcal{S}$ is closed under asymptotic equivalence; see, e.g. Theorem 3 of Teugels (1975). Therefore, by relation (4.9), we readily arrive at the following:

**Lemma 4.3** *Let $(X, Y)$ follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. If $F \in \mathcal{S}$ and (b1)–(b4) hold then $H \in \mathcal{S}$.*

For a distribution function $F$ on $\mathbb{R}$, under (b1), (b2) and (b4) it is straightforward to verify that the relation

$$\overline{F}\left(x - cb(x)\right) \sim \overline{F}(x) \tag{4.10}$$

holds for all $c > 0$. The following lemma is inspired by the work of Zhou et al. (2011):

**Lemma 4.4** *Let $(X, Y)$ follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. Under (b1)–(b4), it holds for all $c > 0$ that*

$$\overline{H}\left(x - cb(x)\right) \sim \overline{H}(x). \tag{4.11}$$

**Proof.** We only need to prove (4.11) with $c = 1$. For arbitrarily fixed $0 < \varepsilon, \delta < 1$, by (4.9) we have

$$\begin{aligned}
\overline{H}\left(x - b(x)\right) &\sim \Pr\left(XZ_\theta > x - b(x)\right) \\
&= \left(\int_0^\varepsilon + \int_\varepsilon^1 + \int_1^{b(x)} + \int_{b(x)}^\infty\right) \overline{F}\left(\frac{x - b(x)}{y}\right) \mathrm{d}G_\theta(y) \\
&= \sum_{i=1}^4 I_i(x).
\end{aligned} \tag{4.12}$$

We deal with the four terms on the right-hand side of (4.12) one by one. By (b1) and relation (4.10) with $c = 1/\varepsilon$, it holds for all large $x$ that

$$\begin{aligned}
I_1(x) &\leq \overline{F}\left(\frac{x-b(x)}{\varepsilon}\right) G_\theta\left((0,\varepsilon]\right) \\
&\leq \overline{F}\left(\frac{x}{\varepsilon}-\frac{1}{\varepsilon}b\left(\frac{x}{\varepsilon}\right)\right) G_\theta\left((0,\varepsilon]\right) \\
&\leq (1+\delta)\, G_\theta\left((0,\varepsilon]\right) \overline{F}\left(\frac{x}{\varepsilon}\right) \\
&= (1+\delta)\, \frac{G_\theta\left((0,\varepsilon]\right)}{\overline{G_\theta}(\varepsilon)} \Pr\left(X > \frac{x}{\varepsilon}, Z_\theta > \varepsilon\right) \\
&\leq (1+\delta)\, \frac{G_\theta\left((0,\varepsilon]\right)}{\overline{G_\theta}(\varepsilon)} \Pr\left(XZ_\theta > x\right).
\end{aligned}$$

Similarly, it holds for all large $x$ that

$$\begin{aligned}
I_2(x) &\leq \int_\varepsilon^1 \overline{F}\left(\frac{x}{y}-\frac{b(x)}{\varepsilon}\right) \mathrm{d}G_\theta(y) \\
&\leq \int_\varepsilon^1 \overline{F}\left(\frac{x}{y}-\frac{1}{\varepsilon}b\left(\frac{x}{y}\right)\right) \mathrm{d}G_\theta(y) \\
&\leq (1+\delta) \int_\varepsilon^1 \overline{F}\left(\frac{x}{y}\right) \mathrm{d}G_\theta(y).
\end{aligned}$$

By (b2) and (b4), it holds for all large $x$ that

$$I_3(x) \leq \int_1^{b(x)} \overline{F}\left(\frac{x}{y}-b\left(\frac{x}{y}\right)\right) \mathrm{d}G_\theta(y) \leq (1+\delta) \int_1^{b(x)} \overline{F}\left(\frac{x}{y}\right) \mathrm{d}G_\theta(y).$$

Finally, by (4.8) and (b3),

$$I_4(x) \leq \overline{G_\theta}\left(b(x)\right) \asymp \overline{G}\left(b(x)\right) = o\left(\overline{H}(x)\right).$$

Substituting these estimates into (4.12) yields that

$$\begin{aligned}
&\overline{H}\left(x-b(x)\right) \\
&\lesssim (1+\delta)\, \frac{G_\theta\left((0,\varepsilon]\right)}{\overline{G_\theta}(\varepsilon)} \Pr\left(XZ_\theta > x\right) + (1+\delta) \int_\varepsilon^{b(x)} \overline{F}\left(\frac{x}{y}\right) \mathrm{d}G_\theta(y) + o\left(\overline{H}(x)\right) \\
&\lesssim (1+\delta) \left(\frac{G_\theta\left((0,\varepsilon]\right)}{\overline{G_\theta}(\varepsilon)} + 1\right) \Pr\left(XZ_\theta > x\right) + o\left(\overline{H}(x)\right) \\
&\sim (1+\delta) \left(\frac{G_\theta\left((0,\varepsilon]\right)}{\overline{G_\theta}(\varepsilon)} + 1\right) \overline{H}(x),
\end{aligned}$$

where in the last step we used (4.9) again. Letting both $\varepsilon \downarrow 0$ and $\delta \downarrow 0$ on the right-hand side above leads to $\overline{H}\left(x-b(x)\right) \lesssim \overline{H}(x)$, which is equivalent to (4.11) with $c = 1$. ■

The following lemma is well known and can be found in Embrechts and Goldie (1980), Cline (1986, Corollary 1) and Tang and Tsitsiashvili (2003, Lemma 3.2):

**Lemma 4.5** *Let $F_1$ and $F_2$ be two distribution functions on $\mathbb{R}$ and let $F = F_1 * F_2$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$ and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F \in \mathcal{S}$ and $\overline{F}(x) \sim \overline{F_1}(x) + \overline{F_2}(x)$.*

**Proof of Theorem 3.1**: As mentioned in the beginning of this section, we only need to prove that relation (4.3) holds and the distribution function of $T_n$ belongs to the class $\mathcal{S}$. We proceed by induction on $n$. Note that:

(1:1) $\overline{G}\left(b(x)\right) = o\left(\overline{H}(x)\right)$ and $\overline{H}\left(x - b(x)\right) \sim \overline{H}(x)$ (by Lemma 4.4),
(1:2) relation (4.3) trivially holds for $n = 1$,
(1:3) $T_1 = X_1 Y_1$ follows a subexponential distribution function (by Lemma 4.3).

Now assume that:

(n:1) $\overline{G}\left(b(x)\right) = o\left(\overline{H_n}(x)\right)$ and $\overline{H_n}\left(x - b(x)\right) \sim \overline{H_n}(x)$,
(n:2) relation (4.3) holds for $n$,
(n:3) both $X_n \prod_{j=1}^{n} Y_j$ and $T_n$ follow subexponential distribution functions.

We are going to prove all these assertions for $n+1$, namely:

(n+1:1) $\overline{G}\left(b(x)\right) = o\left(\overline{H_{n+1}}(x)\right)$ and $\overline{H_{n+1}}\left(x - b(x)\right) \sim \overline{H_{n+1}}(x)$,
(n+1:2) relation (4.3) holds for $n+1$,
(n+1:3) both $X_{n+1} \prod_{j=1}^{n+1} Y_j$ and $T_{n+1}$ follow subexponential distribution functions.

First of all, note that $T_n$ and $X_{n+1}$ are independent of each other, that, by (n:3), both of them follow subexponential distribution functions (hence are long tailed), and that, by (n:2), if $\overline{G}(1) > 0$ then $\Pr\left(X_{n+1} > x\right) = O\left(\Pr\left(T_n > x\right)\right)$, while if $\overline{G}(1) = 0$ then $\Pr\left(T_n > x\right) = O\left(\Pr\left(X_{n+1} > x\right)\right)$. Thus, by Lemma 4.5,

$$\Pr\left(T_n + X_{n+1} > x\right) \sim \Pr\left(T_n > x\right) + \Pr\left(X_{n+1} > x\right) \sim \sum_{i=1}^{n} \overline{H_i}(x) + \overline{F}(x), \tag{4.13}$$

where in the last step we used (n:2).

Consider (n+1:1). If $\overline{G}(1) > 0$ then

$$\overline{H_{n+1}}(x) \geq \Pr\left(X_{n+1} \prod_{j=1}^{n+1} Y_j > x, Y_1 > 1, \ldots, Y_n > 1\right) \geq \overline{H}(x)\overline{G}(1)^n.$$

Thus, $\overline{G}\left(b(x)\right) = o\left(\overline{H}(x)\right) = o\left(\overline{H_{n+1}}(x)\right)$. If $\overline{G}(1) = 0$ then the relation $\overline{G}\left(b(x)\right) = o\left(\overline{H_{n+1}}(x)\right)$ trivially holds since $\overline{G}\left(b(x)\right) = 0$ for all large $x$. Moreover, note that, since $\overline{H_n}(x - b(x)) \sim \overline{H_n}(x)$ by (n:1), the proof of $\overline{H_{n+1}}\left(x - b(x)\right) \sim \overline{H_{n+1}}(x)$ is similar to but easier than the proof of Lemma 4.4.

Next, consider (n+1:2). Introduce six independent random variables $X^*$, $X_1^*$, $X_2^*$, $Y^*$, $Y_1^*$ and $Y_2^*$ the same as done for Lemma 4.2 and let them be independent of $T_n$. Similarly as done in (4.6),

$$\begin{aligned}
\Pr\left(T_{n+1} > x\right) &= \Pr\left(\left(T_n + X_{n+1}\right) Y_{n+1} > x\right) \\
&= (1+\theta) \Pr\left(\left(T_n + X^*\right) Y^* > x\right) - \theta \Pr\left(\left(T_n + X_1^* \vee X_2^*\right) Y^* > x\right) \\
&\quad - \theta \Pr\left(\left(T_n + X^*\right)\left(Y_1^* \vee Y_2^*\right) > x\right) + \theta \Pr\left(\left(T_n + X_1^* \vee X_2^*\right)\left(Y_1^* \vee Y_2^*\right) > x\right) \\
&= (1+\theta) J_1(x) - \theta J_2(x) - \theta J_3(x) + \theta J_4(x).
\end{aligned} \tag{4.14}$$

We only deal with the tail probability $J_1(x)$ in detail. By (4.13), for $x > 0$,

$$\begin{aligned} J_1(x) &= \left(\int_0^{b(x)} + \int_{b(x)}^{\infty}\right) \Pr\left(T_n + X^* > \frac{x}{y}\right) \mathrm{d}G(y) \\ &= (1+o(1)) \int_0^{b(x)} \left(\sum_{i=1}^{n} \overline{H_i}\left(\frac{x}{y}\right) + \overline{F}\left(\frac{x}{y}\right)\right) \mathrm{d}G(y) + O\left(\overline{G}\left(b(x)\right)\right) \\ &\sim \int_0^{\infty} \left(\sum_{i=1}^{n} \overline{H_i}\left(\frac{x}{y}\right) + \overline{F}\left(\frac{x}{y}\right)\right) \mathrm{d}G(y), \end{aligned}$$

where we used $\overline{G}\left(b(x)\right) = o(1)\int_0^{\infty} \overline{F}\left(x/y\right) \mathrm{d}G(y)$ due to (b3) and relations (4.4) and (4.7). In a similar way with some obvious modifications, we have the following:

$$\begin{aligned} J_2(x) &\sim \int_0^{\infty} \left(\sum_{i=1}^{n} \overline{H_i}\left(\frac{x}{y}\right) + 2\overline{F}\left(\frac{x}{y}\right)\right) \mathrm{d}G(y), \\ J_3(x) &\sim \int_0^{\infty} \left(\sum_{i=1}^{n} \overline{H_i}\left(\frac{x}{y}\right) + \overline{F}\left(\frac{x}{y}\right)\right) \mathrm{d}G^2(y), \\ J_4(x) &\sim \int_0^{\infty} \left(\sum_{i=1}^{n} \overline{H_i}\left(\frac{x}{y}\right) + 2\overline{F}\left(\frac{x}{y}\right)\right) \mathrm{d}G^2(y). \end{aligned}$$

Substituting these estimates into (4.14) then using (4.8) and (4.9), we obtain

$$\begin{aligned} \Pr\left(T_{n+1} > x\right) &\sim \int_0^{\infty} \sum_{i=1}^{n} \overline{H_i}\left(\frac{x}{y}\right) \mathrm{d}G(y) + (1-\theta)\int_0^{\infty} \overline{F}\left(\frac{x}{y}\right) \mathrm{d}G(y) + \theta \int_0^{\infty} \overline{F}\left(\frac{x}{y}\right) \mathrm{d}G^2(y) \\ &= \sum_{i=2}^{n+1} \overline{H_i}(x) + \int_0^{\infty} \overline{F}\left(\frac{x}{y}\right) \mathrm{d}G_{\theta}(y) \\ &\sim \sum_{i=1}^{n+1} \overline{H_i}(x), \end{aligned} \tag{4.15}$$

showing that relation (4.3) still holds for $n+1$.

Finally, consider (n+1:3). Note that $H_n \in \mathcal{S}$ by (n:3) and that $\overline{G}\left(b(x)\right) = o\left(\overline{H_n}(x)\right)$ and $\overline{H_n}\left(x - b(x)\right) \sim \overline{H_n}(x)$ by (n:1). Thus, applying Lemma 4.1 we obtain $H_{n+1} \in \mathcal{S}$. In other words, $H_i \in \mathcal{S} \subset \mathcal{L}$ for all $i = 1, \ldots, n+1$. Hence by (4.15), $T_{n+1}$ is long tailed. Moreover, by (4.15) again, if $\overline{G}(1) > 0$ then $\Pr\left(T_{n+1} > x\right) \asymp \overline{H_{n+1}}(x)$, while if $\overline{G}(1) = 0$ then $\Pr\left(T_{n+1} > x\right) \asymp \overline{H}(x)$. Therefore, for both cases, applying Theorem 2.1(a) of Klüppelberg (1988) we obtain the subexponentiality of $T_{n+1}$. This ends the proof of Theorem 3.1.

## 4.2 Proof of Theorem 3.2

The following two lemmas are restatements of Lemma 3.2 and Theorem 2.1 of Tang (2006b), respectively:

**Lemma 4.6** *For two distribution functions $G$ and $H$ with $\overline{G}(x) > 0$ and $\overline{H}(x) > 0$ for all $x \in \mathbb{R}^+$, the relation $\overline{G}(cx) = o\left(\overline{H}(x)\right)$ holds for all $c > 0$ if and only if there is a function $b(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that (b1)–(b3) hold, namely:*

*(b1)* $b(x) \uparrow \infty$,

*(b2)* $b(x)/x \downarrow 0$, *and*

*(b3)* $\overline{G}\left(b(x)\right) = o\left(\overline{H}(x)\right)$.

**Lemma 4.7** *Let $X$ and $Y$ be two independent random variables with distribution functions $F$ on $\mathbb{R}$ and $G$ on $\mathbb{R}^+$, respectively. Denote by $H$ the distribution function of their product $XY$. If $F \in \mathcal{A}$ and $\overline{G}(cx) = o\left(\overline{H}(x)\right)$ for all $c > 0$ then $H \in \mathcal{A}$.*

Recall relations (4.8) and (4.9) under the conditions of Lemma 4.2. Thus, applying Lemma 4.7, we can easily obtain an analogue of Lemma 4.3 as follows:

**Lemma 4.8** *Let $(X, Y)$ follow a bivariate FGM distribution function (1.3) with $\theta \in (-1, 1]$. If $F \in \mathcal{A}$ and (b1)–(b3) hold then $H \in \mathcal{A}$.*

**Proof of Theorem 3.2**: As mentioned before, we only need to prove that relation (4.3) holds and the distribution function of $T_n$ belongs to the class $\mathcal{A}$. We still proceed by induction on $n$. Note that:

(1:1) $\overline{G}\left(b(x)\right) = o\left(\overline{H}(x)\right)$,

(1:2) relation (4.3) trivially holds for $n = 1$,

(1:3) $T_1 = X_1 Y_1$ follows a distribution function in the class $\mathcal{A}$ (by Lemma 4.8).

Now assume that:

(n:1) $\overline{G}\left(b(x)\right) = o\left(\overline{H_n}(x)\right)$,

(n:2) relation (4.3) holds for $n$,

(n:3) both $X_n \prod_{j=1}^{n} Y_j$ and $T_n$ follow distribution functions in the class $\mathcal{A}$.

We are going to prove all these assertions for $n + 1$, namely:

(n+1:1) $\overline{G}\left(b(x)\right) = o\left(\overline{H_{n+1}}(x)\right)$,

(n+1:2) relation (4.3) holds for $n + 1$,

(n+1:3) both $X_{n+1} \prod_{j=1}^{n+1} Y_j$ and $T_{n+1}$ follow distribution functions in the class $\mathcal{A}$.

The same as in the proof of Theorem 3.1, the relations in (4.13) hold. For (n+1:1), the verification of $\overline{G}\left(b(x)\right) = o\left(\overline{H_{n+1}}(x)\right)$ is the same as in the proof of Theorem 3.1. For (n+1:2), recalling Lemma 4.6, by relations (4.3) and (4.13) for $n$, the derivations of (4.14)–(4.15) are still valid, showing that relation (4.3) holds for $n + 1$.

Finally, consider (n+1:3). Note that $H_n \in \mathcal{A}$ by (n:3) and that $\overline{G}\left(b(x)\right) = o\left(\overline{H_n}(x)\right)$ by (n:1). Thus, applying Lemmas 4.6 and 4.7 we obtain $H_{n+1} \in \mathcal{A}$. In other words, $H_i \in \mathcal{A} \subset \mathcal{S}$ for all $i = 1, \ldots, n + 1$. Then, similarly as in the proof of Theorem 3.1, we obtain the subexponentiality of $T_{n+1}$. Furthermore, by definition, $H_i \in \mathcal{A}$ implies that there is some constant $c_i > 1$ such that

$$\limsup_{x\to\infty} \frac{\overline{H_i}(c_i x)}{\overline{H_i}(x)} < 1.$$

With $c = \max\{c_1, c_2, \dots, c_{n+1}\} > 1$, by (4.15) we have

$$\begin{aligned}\limsup_{x\to\infty} \frac{\Pr(T_{n+1} > cx)}{\Pr(T_{n+1} > x)} &\le \limsup_{x\to\infty} \frac{\sum_{i=1}^{n+1} \overline{H_i}(c_i x)}{\sum_{i=1}^{n+1} \overline{H_i}(x)} \\ &\le \max_{1\le i\le n+1} \limsup_{x\to\infty} \frac{\overline{H_i}(c_i x)}{\overline{H_i}(x)} \\ &< 1.\end{aligned}$$

Therefore, the distribution function of $T_{n+1}$ belongs to the class $\mathcal{A}$. This ends the proof of Theorem 3.2.

## 4.3 Proof of Corollary 3.1

Copied below is the well-known Breiman's theorem; see Breiman (1965) and Cline and Samorodnitsky (1994):

**Lemma 4.9** *Let $X$ and $Y$ be two independent random variables with distribution functions $F$ on $\mathbb{R}$ and $G$ on $\mathbb{R}^+$, respectively. Denote by $H$ the distribution function of their product $XY$. If $F \in \mathrm{MDA}(\Phi_\alpha)$ for some $\alpha > 0$ and $\mathrm{E}[Y^{\alpha+\varepsilon}] < \infty$ for some $\varepsilon > 0$ then*

$$\lim_{x\to\infty} \frac{\overline{H}(x)}{\overline{F}(x)} = \mathrm{E}[Y^\alpha].$$

The following lemma forms the main ingredient of the proof of Corollary 3.1:

**Lemma 4.10** *We have the following:*

*(i) Under the conditions of Corollary 3.1(i), it holds for each $i \in \mathbb{N}$ that*

$$\overline{H_i}(x) \sim \left((1-\theta)\mathrm{E}[Y^\alpha] + \theta\mathrm{E}[(Y_1^* \vee Y_2^*)^\alpha]\right) \left(\mathrm{E}[Y^\alpha]\right)^{i-1} \overline{F}(x),$$

*where $Y_1^*$ and $Y_2^*$ are two independent random variables identically distributed as $Y$.*

*(ii) Under the conditions of Corollary 3.1(ii), it holds for each $i \in \mathbb{N}$ that*

$$\overline{H_i}(x) \sim (1+\theta)\,\overline{F}\left(\frac{x}{\hat{y}^i}\right) \left(\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^{i+1}}{x} a\left(\frac{x}{\hat{y}^i}\right)\right)\right)^i.$$

**Proof.** (i) Apply Lemma 4.9 to relation (4.4) to obtain

$$\lim_{x\to\infty} \frac{\overline{H}(x)}{\overline{F}(x)} = (1-\theta)\mathrm{E}[Y^\alpha] + \theta\mathrm{E}[(Y_1^* \vee Y_2^*)^\alpha]; \tag{4.16}$$

see also Theorem 2.1(i) of Jiang and Tang (2011) for a result more general than (4.16). Thus, the distribution function $H$ of $X_i Y_i$ belongs to the class $\mathrm{MDA}(\Phi_\alpha)$. Then, for $i = 2, 3, \dots$, by Lemmas 4.9 and relation (4.16), in turn,

$$\begin{aligned}\overline{H_i}(x) &= \Pr\left(X_i Y_i \prod_{j=1}^{i-1} Y_j > x\right) \\ &\sim \left(\mathrm{E}[Y^\alpha]\right)^{i-1} \overline{H}(x) \\ &\sim \left((1-\theta)\mathrm{E}[Y^\alpha] + \theta\mathrm{E}[(Y_1^* \vee Y_2^*)^\alpha]\right) \left(\mathrm{E}[Y^\alpha]\right)^{i-1} \overline{F}(x).\end{aligned}$$

(ii) Note that every distribution function $F \in \mathrm{MDA}(\Lambda)$ with an upper endpoint $\hat{x} = \infty$ is rapidly-varying tailed. Recall Theorem 2.1(ii) of Jiang and Tang (2011), which, in our situation, implies that

$$\lim_{x\to\infty} \frac{\Pr(XY > x)}{\Pr(X^*Y^* > x)} = 1 + \theta, \tag{4.17}$$

where $X^*$, $Y^*$ are two independent random variables with $X^*$ identically distributed as $X$ and $Y^*$ as $Y$, and are independent of $\{(X_n, Y_n), n \in \mathbb{N}\}$. Then by (4.17) and Theorem 3.1(i) of Hashorva et al. (2010), for $i = 1, 2, \ldots$,

$$\begin{aligned}
\overline{H_i}(x) &= \mathrm{E}\left[\Pr\left( X_i Y_i \prod_{j=1}^{i-1} Y_j > x \,\middle|\, Y_1, \ldots, Y_{i-1}\right)\right] \\
&\sim (1+\theta)\Pr\left( X^* Y^* \prod_{j=1}^{i-1} Y_j > x\right) \\
&= (1+\theta)\Pr\left( X^* \prod_{j=1}^{i} \frac{Y_j}{\hat{y}} > \frac{x}{\hat{y}^i}\right) \\
&= (1+\theta)\,\overline{F}\left(\frac{x}{\hat{y}^i}\right)\left(\Gamma(\alpha+1)\overline{G}\left(\hat{y} - \frac{\hat{y}^{i+1}}{x} a\left(\frac{x}{\hat{y}^i}\right)\right)\right)^i .
\end{aligned}$$

This ends the proof of Lemma 4.10. ■

**Proof of Corollary 3.1:** For item (i), by (4.16) it is easy to verify that (b1)–(b3) hold with $b(x) = x^{1-\delta}$ for some $0 < \delta < \varepsilon/(\alpha + \varepsilon)$; see also Lemma 3.7 of Tang and Tsitsiashvili (2003). For item (ii), (b1)–(b3) automatically hold because $Y$ is bounded above. Hence, Theorem 3.2 is applicable for both items (i) and (ii). Then we apply Lemma 4.10 to (3.1) to complete the proof of Corollary 3.1.

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