Asymptotic Behaviour of Extinction Probability of Interacting Branching Collision Processes

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Abstract

Although the exact expressions for the extinction probabilities of the Interacting Branching Collision Processes (IBCP) are very recently given by Chen et al [4], some of these expressions are very complicated and hence useful information regarding asymptotic behaviour, for example, is hardly to be obtained. Also, these exact expressions take very different forms for different cases and thus seem lacking of homogeneity. In this paper, we show that the asymptotic behaviour of these extremely complicated and tangled expressions for extinction probabilities of IBCP follows an elegant and homogenous power law which takes a very simple form. In fact we are able to show that if the extinction is not certain then the extinction probabilities $\{a_n\}$ follows an harmonious and simple asymptotic law of $a_n \sim kn^{-\alpha}\rho_c^n$ as $n \to \infty$ where k and α are two constants and ρ_c is the unique positive zero of the $C(s)$ and $C(s)$ is the generating function of the infinitesimal collision rates.
Moreover, the interesting and important quantity α takes a very simple and uniform form which could be interpreted as the "spectrum", ranging from minus infinity to positive infinity, of the interaction between the two components of branching and collision of the Interacting Branching Collision Processes.

Keywords: Markov branching processes; Interacting branching collision processes; Extinction Probability; Asymptotic behaviour.

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1. Introduction

Due to the urgent need in analyzing practical models and developing corresponding challenging mathematical theory, the focus of research interests on branching models has been shifted from the independent Markov Branching Processes (MBP) into interacting branching systems and thus the latter has been attracted more and more extensive research attention. Many new interacting branching models have been posted and analyzed and the corresponding theory is also fast developing. For the traditional independent Markov Branching Processes, the good references are, among many others, Harris [7], Athreya and Ney [3], and Asmussen and Hering [1] and the huge references therein while the references for interacting branching models could be found in, for example, Sevast'yanov [14], Otter [13], Sevast'yanov and Kalinkin [15], Athreya and Jagers [2], Kalinkin [8, 9, 10], Li and Chen [12], Chen et al [5, 6], and Lange [11].

Very recently, Chen et al [4] considered an important and challenging model of interacting branching system, the so-called Interacting Branching Collision Process (IBCP) which consists of two strongly interacting components: the branching component and the collision component. Basic properties on uniqueness and extinction probabilities have been discussed and many important results have been obtained. In particular, they proved that there exists only one IBCP which is just the Feller minimal process for any given infinitesimal generator, the so-called q-matrix Q. They also obtained an "if and only if" condition under which the IBCP would go to extinction with probability one and revealed all kinds of exact expressions of extinction probabilities when the extinction is not certain.

However, though given, some of these exact expressions are very complicated and it is difficult, for example, to obtain useful information about the asymptotic behaviour of the extinction probability from these exact expressions. The intuitive meanings of these complex extinction probabilities are also unclear. These disadvantages limit the applications of these obtained results in practical models. Hence revealing simple asymptotic behaviour for these complex expressions is of great significance.

The basic aim of this paper is therefore to reveal the asymptotic behaviour for these complex expressions of extinction probabilities. We shall show that the asymptotic behaviour for these complicated extinction probabilities take a very simple form.

This short paper has only three sections. In the following Section 2, we report the main conclusions obtained in this paper. The proofs of these conclusions are given in the last Section 3. Comparing with the previous two sections, this last Section 3 is a little bit lengthy which is, in fact, necessary. Indeed, a few of separated theorems, dealing with different cases, are given in Section 3 in order to show the harmonious power law stated in Section 2. In the end of this paper, we use a simple example to demonstrate our elegant results.

2. Main Results

Following Chen et al [4], we define an Interacting Branching-Collision Process (henceforth referred to as an IBCP) as a continuous-time Markov chain on the state space $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ satisfies $P'(t) = P(t)Q$ where the interacting branching-collision infinitesimal q-matrix (henceforth referred to as an IBC q-matrix) $Q =$

$(q_{ij};\ i,j \in \mathbf{Z}_+)$ is given by

$$q_{ij} = \begin{cases} \binom{i}{2} c_{j-i+2} + i b_{j-i+1}, & \text{if } i \geq 1, j \geq i-2 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where

$$\begin{cases} c_0 > 0, c_j \geq 0\ (j \neq 2), \sum_{k=3}^{\infty} c_k > 0,\ 0 < \sum_{j \neq 2} c_j = -c_2 < \infty \\ b_0 > 0, b_j \geq 0\ (j \neq 1), \sum_{k=2}^{\infty} b_k > 0,\ 0 < \sum_{j \neq 1} b_j = -b_1 < \infty. \end{cases} \tag{2.2}$$

together with the conventions $b_{-1} = 0$ and $\binom{1}{2} = 0$. In order to avoid discussing some degenerated and thus trivial cases, we also assume, through this paper, that $\sum_{k=0}^{\infty} c_{2k+1} \neq 0$.

Again, following Chen et al [4], we define the generating functions of the two known sequences $\{c_k; k \geq 0\}$ and $\{b_k; \geq 0\}$ as

$$C(s) = \sum_{k=0}^{\infty} c_k s^k \quad \text{and} \quad B(s) = \sum_{k=0}^{\infty} b_k s^k. \tag{2.3}$$

By Theorem 3.2 in Chen et al [4] we know that for any given IBC q-matrix Q, there exists only one IBCP. Now, let $\{X(t); t \geq 0\}$ denote this unique IBCP with the given IBC q-matrix Q as defined in (2.1) – (2.2) and let $P(t) = (p_{ij}(t); i,j \in \mathbf{Z}_+)$ be the Q-function of this unique IBCP. Let also

$$\tau_0 = \inf\{t > 0; X(t) = 0\}$$

and

$$a_i = P(\tau_0 < \infty | X(0) = i), \quad i \geq 1,$$

be the extinction time and extinction probability, respectively.

Then the following conclusions were obtained in Chen et al [4].

Proposition 2.1. (i) *The equation $C(s) = 0$ has either two roots or three roots in the complex disk $\{z; |z| \leq 1\}$ and all these roots are real. More specifically, if $C'(1) \leq 0$ then $C(s) > 0$ for all $s \in [0,1)$ and 1 is the only root of the equation $C(s) = 0$ in $[0,1]$, which is simple or with multiplicity 2 according to $C'(1) < 0$ or $C'(1) = 0$, while if $0 < C'(1) \leq \infty$ then $C(s) = 0$ has an additional simple root ρ_c satisfying $0 < \rho_c < 1$ such that $C(s) > 0$ for $s \in (0, \rho_c)$ and $C(s) < 0$ for $s \in (\rho_c, 1)$. In addition, $C(s) = 0$ has exactly one root, denoted by ξ_c, in $[-1, 0]$ such that $C(s) > 0$ for all $s \in (\xi_c, 0]$ and $|\xi_c| < \rho_c$. Moreover, $C(z) = 0$ has no other root in the complex disk $\{z; |z| \leq 1\}$.*

(ii) *The equation $B(s) = 0$ has either one root or two roots in the complex disk $\{z; |z| \leq 1\}$ and all these roots are positive. More specifically, if $B'(1) \leq 0$ then $B(s) > 0$ for all $s \in [-1, 1)$ and 1 is the only root of $B(s) = 0$ in $[0,1]$. If $0 < B'(1) \leq +\infty$ then $B(s) = 0$ has an additional root in $[0,1)$, denoted by ρ_b, such that $B(s) > 0$ for all $s \in [-1, \rho_b)$ and $B(s) < 0$ for $s \in (\rho_b, 1)$. Moreover, $B(z) = 0$ has no other root in the complex disk $\{z; |z| \leq 1\}$.*

Throughout this paper, we shall let ρ_c and ρ_b denote the smallest nonnegative root of $C(s) = 0$ and $B(s) = 0$ respectively.

Proposition 2.2. *Suppose that Q is an IBC q-matrix as defined in (2.1) − (2.2) and let $P(t) = (p_{ij}(t); i, j \geq 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \geq 0)$ be the (in fact, unique, see the following Proposition 2.3) Q-function and its Q-resolvent that satisfy the Kolmogorov forward equations, respectively. Then for any $i \geq 0, t \geq 0, \lambda > 0$ and $|s| < 1$, we have*

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{C(s)}{2} \cdot \frac{\partial^2 F_i(t,s)}{\partial s^2} + B(s) \cdot \frac{\partial F_i(t,s)}{\partial s} \tag{2.4}$$

and

$$\Phi_i(\lambda, s) - s^i = \frac{C(s)}{2} \cdot \frac{\partial^2 \Phi_i(\lambda,s)}{\partial s^2} + B(s) \cdot \frac{\partial \Phi_i(\lambda,s)}{\partial s} \tag{2.5}$$

where $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$ and $\Phi_i(\lambda, s) = \sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j$.

Proposition 2.3.

(i) For any IBC q−matrix Q, there exists only one IBCP which is the Feller minimal Q−process. Moreover, this unique IBCP is honest (i.e. Q is regular) if and only if $C'(1) \leq 0$.

(ii) The extinction probability of this unique IBCP, starting from state $i \geq 1$, is 1 if and only if either

(a) $C'(1) \leq 0$ and $B'(1) \leq 0$
or

(b) $C'(1) \leq 0$, $0 < B'(1) \leq \infty$ and $J = \int_{\xi_c}^1 \frac{A(y)}{C(y)} \mathrm{d}y = +\infty$ (equivalently, $J_0 = \int_0^1 \frac{A(y)}{C(y)} \mathrm{d}y = +\infty$) where

$$A(s) = \exp\{\int_0^s \frac{2B(x)}{C(x)} \mathrm{d}x\}. \tag{2.6}$$

It should be noticed that throughout this paper, we shall use (a_n) to denote the extinction probability of IBCP when the process stars at state n which has no relationship with $A(s)$ defined in the above (2.6). In particular, (a_n) is not the n'th coefficient of the Taylor series expansion of the function $A(s)$.

Based on the above propositions, Chen et al [4] further deeply investigated the extinction probability. In particular, they proved that if $C'(1) < 0$ and $0 < B'(1) < \infty$ (and thus Q is regular) then the extinction probabilities $a_i = 1$ is true. They further showed that if $C'(1) = 0$ and $0 < B'(1) < \infty$ (and thus Q is still regular) then both the cases of $a_i < 1$ ($\forall i \geq 1$) and $a_i \equiv 1$ ($\forall i \geq 1$) may occur and when $a_i < 1$, the exact expressions for a_i ($i \geq 1$) are given. If $C'(1) > 0$ (and thus Q is irregular by Proposition 2.3), then $a_i < 1$ ($\forall i \geq 1$) is certain and all kinds of expressions for the extinction probabilities a_i ($i \geq 1$) are given in Chen et al [4].

However, although the explicit expressions for extinction probabilities for IBCP are given in Chen et al [4], these expressions are sometimes extremely complicated, see their Theorem

5.8, for example. It seems very hard to draw useful information from these very complicated expressions. For example, we could know little about the asymptotic behaviour of these extinction probabilities to which we are particularly interested in. Another disadvantage for these expressions is that their forms look extremely different for different cases and thus the deep relationships among these expressions seem very vague.

The main aim of this paper is therefore to investigate the simple forms of the asymptotic behaviour of these extinction probabilities in order to overcome the above mentioned shortcomings. Of course, in investigating asymptotic behaviour, we are only interested in the case of $a_i < 1$ since otherwise the question would be trivial. Hence in discussing asymptotic behaviour, we are only interested in, by Proposition 2.3, two cases: either $0 < C'(1) \leq \infty$ or $C'(1) = 0$ and $J_0 = \int_0^1 \frac{A(y)}{C(y)} \mathrm{d}y < \infty$. For the latter case, we shall further assume $B'(1) < \infty$ since for this latter case we are less interested in the uninformative situation of $B'(1) = \infty$.

Surprisingly, we shall show that as the asymptotic behaviour is concerned, the extinction probabilities of the IBCP display an extremely simple and harmonic feature. Indeed, the asymptotic behaviour of the extinction probabilities just follows a simple power law and, moreover, different situations, to which the exact expressions for extinction probabilities are very complicated and extremely different as mentioned above, are just referring to a constant value, see the following Remark 3.11 together with Remark 3.12.

Our main results obtained in this short paper are the following two conclusions which deal with two different cases: the $q-$matrix Q is regular or irregular.

Theorem 2.4. *Suppose $C'(1) = 0, C''(1) < \infty, B'(1) < \infty$ and $J_0 = \int_0^1 \frac{A(y)}{C(y)} \mathrm{d}y < \infty$ and thus $\rho_c = 1$ (and hence the q-matrix Q is regular). Then as $n \to \infty$, we have*

$$a_n \sim k n^{-\alpha} \qquad (n \to \infty) \tag{2.7}$$

where $\alpha = \frac{4B'(1)}{C''(1)} - 1 > 0$ and k is a constant which is independent of n.

Theorem 2.5. *Suppose $0 < C'(1) \leq +\infty$ and thus $0 < \rho_c < 1$ (and hence the q-matrix Q is irregular). Further assume that $\rho_b \neq \rho_c$. Then as $n \to \infty$,*

$$a_n \sim k n^{-\alpha} \rho_c^n \qquad (n \to \infty) \tag{2.8}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$ and k is a constant which is independent of n.

We see that form (2.8) is extremely simple and harmonic. If one compares (2.8) with the complicated expressions given in Theorems 5.6 to 5.10 of Chen et al [4], one would feel that they would have discussed totally different problems. Note that in Theorem 2.5, we have assumed that $\rho_b \neq \rho_c$. This is because if $\rho_b = \rho_c$, then $a_n = \rho_c^n$, see Theorem 5.1 of Chen et al [4], and thus the asymptotic behaviour for the extinction probabilities $\{a_n\}$ is trivial. However, it is easily seen that even for $\rho_c = \rho_b$, (2.8) is still true, since for this case we have $\alpha = 0$ and $k = 1$. For more detailed explanation, see the Remark 3.11 later.

3. Proofs of the Main Results

As a preparation, we first provide the following simple lemma which describes some simple but useful properties of the function $A(s)$ which is defined in (2.6).

Recall by Proposition 2.1, we know that the generating function $C(s) = \sum_{j=0}^{\infty} c_j s^i$ has a negative zero $-1 < \xi_c < 0$ and a smallest positive zero $0 < \rho_c \le 1$ and, furthermore, $\rho_c < 1$ if and only if $0 < C'(1) \le +\infty$.

Lemma 3.1. *The function $A(s)$ defined in (2.6) possesses the following properties.*

(i) $A(y) \sim l(y - \xi_c)^\beta$ as $y \to \xi_c^+$, where $0 < l < \infty$ is a constant and $\beta = \frac{2B(\xi_c)}{C'(\xi_c)}$.

(ii) Suppose $0 < C'(1) \le \infty$ and thus $\rho_c < 1$. Then

$$A(y) \sim l(\rho_c - y)^\alpha \qquad \text{as } y \to \rho_c^-.$$

where $0 < l < \infty$ is a constant (i.e. independent of y) and $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$

(iii) Suppose $C'(1) = 0$, and $C''(1) < 4B'(1) < \infty$. Then

$$A(y) \sim l(1 - y)^\gamma \qquad \text{as } y \to 1^-$$

where $0 < l < \infty$ is a constant (i.e. independent of y) and $\gamma = \frac{4B'(1)}{C''(1)} > 1$.

Proof. We first prove (ii). By Proposition 2.1, we know that the condition $0 < C'(1) \le \infty$ implies that $\rho_c < 1$ is a single zero of $C(s)$ and thus if we let

$$g(x) = \frac{2B(x)(\rho_c - x)}{C(x)} \tag{3.1}$$

then $g(x)$, as a complex function of x, has only one negative zero ξ_c in the open unit disk $\{z; |z| < 1\}$. In particular, $g(x)$ is analytic on the open disk $\{z; |z| < |\xi_c|\}$ and thus could be expanded as a power series of x on the interval $[0, \rho_c)$. Note that in the latter we have viewed $g(x)$ as a real valued function of x. Suppose the expansion takes the form of

$$g(x) = \sum_{k=0}^{\infty} g_k x^k \tag{3.2}$$

where $g_k = \frac{g^{(k)}(0)}{k!}$. By (3.1) and (3.2), we have for $0 < y < \rho_c$

$$\begin{aligned} & \int_0^y \frac{2B(x)}{C(x)} \mathrm{d}x = \int_0^y \frac{g(x)}{\rho_c - x} \mathrm{d}x = \sum_{k=0}^{\infty} g_k \int_0^y \frac{x^k}{\rho_c - x} \mathrm{d}x \\ = & \left(\sum_{k=0}^{\infty} g_k \rho_c^k \right) \int_0^y \frac{\mathrm{d}x}{\rho_c - x} + \sum_{k=1}^{\infty} g_k \sum_{m=1}^{k} (-1)^m \binom{k}{m} \rho_c^{k-m} \int_0^y (\rho_c - x)^{m-1} \mathrm{d}x \\ = & J_1 + J_2 \end{aligned} \tag{3.3}$$

where the meaning of J_1 and J_2 should be self-explained.

By noting (3.2), we see that J_1 in (3.3) is just

$$J_1 = \left(\sum_{k=1}^{\infty} g_k \rho_c^k\right) \int_0^y \frac{\mathrm{d}x}{\rho_c - x} = g(\rho_c) \int_0^y \frac{\mathrm{d}x}{\rho_c - x}$$

where $g(\rho_c) = \lim_{x\to\rho_c^+} \frac{2B(x)(\rho_c - x)}{C(x)} = -\frac{2B(\rho_c)}{C'(\rho_c)}$ which is finite.

Similarly, after some trivial algebra, J_2 in (3.3) can be written as

$$J_2 = \sum_{k=1}^{\infty} g_k \rho_c^k \sum_{m=1}^{k} (-1)^m \frac{\binom{k}{m}}{m} - \sum_{k=1}^{\infty} g_k \rho_c^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \left(\frac{y}{\rho_c} - 1\right)^m \tag{3.4}$$

We recognize that the first term on the right hand side of (3.4) is just a constant that is independent of y and the second term on the right hand side of (3.4) is just a rational function of y and thus is a bounded function of y on $[0, \rho_c]$. It follows from the mean-valued theorem that J_2 can be written as a constant k_1, say, as $y \to \rho_c^-$.

Therefore we obtain that there exists a constant k, such that

$$A(y) = \exp\left\{\int_0^y \frac{2B(x)}{C(x)} \mathrm{d}x\right\} \sim k\,(\rho_c - y)^\alpha \qquad \text{as } y \to \rho_c^-, \tag{3.5}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$. The proof of (ii) is completed.

The proof of (i) is similar as in proving (ii) by just noting that ξ_c is also a single zero of $C(s)$ and $0 < B(\xi_c) < \infty$.

We now prove (iii). Since $C'(1) = 0$ we know from Proposition 2.1 that $C(s)$ has no zero on $[0, 1)$ and 1 is the zero of $C(s)$ with multiplicity 2. Also, since $0 < B'(1) < \infty$ we know that 1 is also a single zero of $B(s)$. It follows that if we let

$$g_1(x) = \frac{2B(x)(1 - x)}{C(x)}$$

then by the same reasoning as in proving (ii), we get that $g_1(x)$ can also be expanded as a power series of x on $[0, 1)$. Now using the similar arguments as in proving (ii) above and noting the fact that

$$\lim_{s\to 1^+} g_1(x) = \frac{-4B'(1)}{C''(1)},$$

it is then easily shown that (iii) is true.

Remark 3.2. Conclusion (iii) in Lemma 3.1 has been essentially proved in Chen et al [4], see the proof of their Corollary 4.3 under the further assumption that $B''(1) < +\infty$. However, it is easily seen that their proof does not depend on this latter assumption and thus this assumption can be removed.

In the following we shall constantly use the following simple and well-known analytic lemma whose proof can be found in any standard textbooks of analysis.

Lemma 3.3. *For any complex number a we have*

$$\lim_{z\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}z^{-a}=1 \tag{3.6}$$

so long as $\mathcal{R}(a)>0$ where $\mathcal{R}(a)$ denotes the real part of the complex a and $\Gamma(\cdot)$ is the gamma function.

In our later application of Lemma 3.3 we actually will only meet the case that the complex number a is just a real (and thus must be a positive) number.

We are now ready to prove our main results of this paper stated in Section 2. For the purpose of giving some further useful information regarding the asymptotic behaviour of the extinction probability than that stated in Theorems 2.4 and 2.5, we shall state and prove the following main conclusions, separately.

Theorem 3.4. *If $C'(1)=0$ and $B'(1)<\infty$, then the extinction probability $\{a_n\}$, starting from state $n\geq 1$, is less than 1 (for all $n\geq 1$) if and only if $C''(1)<4B'(1)$. Moreover, if $C''(1)<4B''(1)$ is satisfied, then*

$$a_n\sim k_1n^{-\alpha}+k_2n^{-\beta}\xi_c^n \qquad as\ n\to\infty \tag{3.7}$$

where k_1 and k_2 are two constants and $\alpha=\frac{4B'(1)}{C''(1)}-1>0$, and $\beta=\frac{2B(\xi_c)}{C'(\xi_c)}>0$.

Furthermore we have

$$a_n\sim kn^{-\alpha} \qquad as\ n\to\infty \tag{3.8}$$

where k is a constant and $\alpha=\frac{4B'(1)}{C''(1)}-1>0$.

Proof. The first part of the conclusion has been proved in Chen et al [4] and thus we turn to prove (3.7) first. Now suppose $C'(1)=0$ and $C''(1)<4B'(1)<\infty$, then by Theorem 4.2 and Corollary 4.3 of Chen at al [4], we know that the extinction probability $\{a_n\}$ starting from $n\geq 1$, is given by

$$a_n=\frac{1}{J}\int_{\xi_c}^{1}\frac{y^nA(y)}{C(y)}\mathrm{d}y \quad (n\geq 1) \tag{3.9}$$

where $J=\int_{\xi_c}^{1}\frac{A(y)}{C(y)}\mathrm{d}y$ is a finite constant which is independent of n. In order to obtain (3.7), we consider the two integrals $I_1^{(n)}=\int_0^1\frac{y^nA(y)}{C(y)}\mathrm{d}y$ and $I_2^{(n)}=\int_{\xi_c}^0\frac{y^nA(y)}{C(y)}\mathrm{d}y$.

Note that under our condition stated in this theorem, we know that for any $n\geq 1$ and $0<\varepsilon<1$, the function $\frac{y^nA(y)}{C(y)}$ is bounded on $[0,\varepsilon]$ and thus we only need to consider the behaviour of $I_1^{(n)}$ when $y\to 1^-$. Noting that $C'(1)=0$ and $C''(1)<\infty$ and thus the function $\frac{C(y)}{(1-y)^2}$ is bounded on $[0,1]$. This fact together with the conclusion (iii) in Lemma 3.1, implies that we may write

$$I_1^{(n)}=c\cdot\int_0^1\frac{y^n(1-y)^\gamma}{(1-y)^2}\mathrm{d}y=c\cdot\int_0^1 y^n(1-y)^{\gamma-2}\mathrm{d}y$$

where $0<c<\infty$ is a constant and $\gamma=\frac{4B'(1)}{C''(1)}>1$ since we have $0<C''(1)<4B'(1)<\infty$. However,

$$\int_0^1 y^n(1-y)^{\gamma-2}\mathrm{d}y=\frac{\Gamma(n+1)\Gamma(\gamma-1)}{\Gamma(n+\gamma)}$$

where $\Gamma(\cdot)$ is the gamma function. Now noting the fact that $\gamma-1>0$, and applying Lemma 3.3 we get that there exists a constant k such that

$$I_1^{(n)} \sim kn^{1-\gamma} \quad (n \to \infty). \tag{3.10}$$

We now turn to consider $I_2^{(n)}$. Similarly as in treating $I_1^{(n)}$ but noting the difference that ξ_c is a single zero of $C(s)$, we can get that, by also using (i) of Lemma 3.1, there exists a constant $\tilde{c}$ such that

$$I_2^{(n)} = \tilde{c} \cdot \int_{\xi_c}^{0} y^n (y-\xi_c)^{\beta-1} \,\mathrm{d}y$$

where $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$. After doing the similar transformation as above, we can get that there exists a constant c such that

$$I_2^{(n)} = c \cdot \xi_c^n \cdot \frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(n+1+\beta)}.$$

Now applying Lemma (3.3) once again and noting $\beta > 0$, we get that there exists a constant $\tilde{k}$ such that

$$I_2^{(n)} \sim \tilde{k} n^{-\beta} \xi_c^n \quad (n \to \infty). \tag{3.11}$$

Combining (3.10) and (3.11) and noting that there exists another constant $1/J$ in the form of $\{a_n\}$, we see that (3.7) is true by simply letting $\alpha = \gamma - 1$ (and hence $\alpha > 0$). Finally, considering $|\xi_c| < 1$, we immediately get (3.8) by using the proven (3.7).

Remark 3.5. We see that (3.8) is the same as (2.7) which we claimed in Theorem 2.4. However, we can see that (3.7) is a finer result than (3.8). Indeed, (3.7) provides some further information than that given in (3.8).

We now turn to consider the more interesting and challenging irregular case, i.e. $0 < C'(1) \le \infty$. Although our initial aim is to prove Theorem 2.5, we shall discuss this case more extensively. The reward is that we can get much more information than that stated in Theorem 2.5. Note that in discussing this irregular case, neither $C'(1) < \infty$ nor $B'(1) < \infty$ is assumed. In other words, we shall cover all possible cases, even if both $C'(1)$ and $B'(1)$ are infinity.

By Proposition 2.3, we know that the condition $0 < C'(1) \le \infty$ implies that $\rho_c < 1$ where ρ_c is the smallest positive zero of $C(s)$. We also know that the generating function $B(s)$ has also the smallest positive zero ρ_b. Hence three relationships between them may occur, i.e. $\rho_b < \rho_c < 1$, $\rho_b = \rho_c < 1$ and $\rho_c < \rho_b \le 1$. However, as asymptotic property of the extinction probability is concerned, the case of $\rho_b = \rho_c < 1$ is trivial since in this case we have $a_n = \rho_c^n$ $(n \ge 1)$. We shall therefore only consider the other two cases. We now first investigate the case of $\rho_b < \rho_c < 1$.

Theorem 3.6. *If $\rho_b < \rho_c < 1$, then the extinction probability of the IBCP, starting from $n \ge 1$, denoted by $\{a_n\}$, possesses the following asymptotic behaviour*

$$a_n \sim k_1 n^{-\alpha} \rho_c^n + k_2 n^{-\beta} \xi_c^n \quad (\textit{as } n \to \infty) \tag{3.12}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$, $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$ *and* k_1 *and* k_2 *are constants which are independent of* n*. Furthermore we have*

$$a_n \sim kn^{-\alpha}\rho_c^n \quad (\text{as } n \to \infty) \tag{3.13}$$

where k *is a constant and* $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$.

Proof. By Theorem 5.3 of Chen et al [4], we know that the extinction probability $\{a_n\}$, starting from $n \geq 1$, is given by

$$a_n = \frac{\int_{\xi_c}^{\rho_c} \frac{y^n A(y)}{C(y)} \mathrm{d}y}{\int_{\xi_c}^{\rho_c} \frac{A(y)}{C(y)} \mathrm{d}y}. \tag{3.14}$$

Since the denominator of the right hand side of (3.14) is just a constant which is independent of n and thus we only need to consider the two integrals $I_1^{(n)} = \int_0^{\rho_c} \frac{y^n A(y)}{C(y)} \mathrm{d}y$ and $I_2^{(n)} = \int_{\xi_c}^0 \frac{y^n A(y)}{C(y)} \mathrm{d}y$.

However, the latter is already analyzed in Theorem 3.4, i.e., (3.11) is still true for our current situation and thus we shall only consider the former. But this is simpler than the case considered in Theorem 3.4 and also very similar as the case of $I_1^{(n)}$ as in Theorem 3.4. Indeed, considering $\rho_c < 1$ is the single zero of $C(s)$ and thus by applying (ii) of Theorem 2.4, we know that there exists a constant k such that

$$I_1^{(n)} = \int_0^{\rho_c} \frac{y^n A(y)}{C(y)} \mathrm{d}y = k \cdot \int_0^{\rho_c} y^n \left(\rho_c - y\right)^{\alpha - 1} \mathrm{d}y$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} > 0$ since both $B(\rho_c)$ and $C'(\rho_c)$ are negative due to the assumption that $\rho_b < \rho_c < 1$.

Now since $\alpha > 0$, we have that

$$\int_0^{\rho_c} y^n \left(\rho_c - y\right)^{\alpha-1} \mathrm{d}y = \rho_c^{n+\alpha} \cdot \int_0^1 x^n (1-x)^{\alpha-1} \mathrm{d}x = \rho_c^{n+\alpha} \frac{\Gamma(n+1)\Gamma(\alpha)}{\Gamma(n+\alpha+1)}$$

and thus by applying Lemma 3.3 once again (since $\alpha > 0$) we obtain that there exists a constant, again denoted by k, such that

$$I_1^{(n)} \sim kn^{-\alpha}\rho_c^n.$$

This together with (3.11) shows that (3.12) is true. Finally, (3.13) follows from (3.14) by noting the fact that $|\xi_c| < \rho_c$. This completes the proof.

Now we turn to consider the more subtle case of $\rho_c < \rho_b \leq 1$. By Proposition 2.1 we know that for this case we have $C'(\rho_c) < 0$ and $B'(\rho_c) > 0$. Following these facts we may face three subcases of $C'(\rho_c) + 2B(\rho_c) < 0$, $C'(\rho_c) + 2B(\rho_c) = 0$ or $C'(\rho_c) + 2B(\rho_c) > 0$. We shall discuss these three subcases separately. We first consider the subcase of $C'(\rho_c) + 2B(\rho_c) = 0$.

Theorem 3.7. *If* $\rho_c < \rho_b \leq 1$ *and* $C'(\rho_c) + 2B(\rho_c) = 0$*, then the extinction probability* $\{a_n\}$*, starting from* $n \geq 1$*, is given by*

$$a_n = \rho_c^n + \sigma n \rho_c^{n-1} \tag{3.15}$$

where $\sigma = -\frac{B(\rho_c)}{B'(\rho_c)}$. *Furthermore,*

$$a_n \sim k n^{-\alpha} \rho_c^n \quad (n \to \infty) \tag{3.16}$$

where $k = \frac{\sigma}{\rho_c}$ *is a constant and* $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} = -1$.

Proof. (3.15) is proved in Theorem 5.5 of Chen et al [4] and then (3.16) follows from (3.15) directly. Also, it is easily seen that the condition $C'(\rho_c) + 2B(\rho_c) = 0$ is equivalent to $\alpha = -1$.

Secondly we consider the subcase of $C'(\rho_c) + 2B(\rho_c) < 0$.

Theorem 3.8. *Suppose* $\rho_c < \rho_b \le 1$ *and* $C'(\rho_c) + 2B(\rho_c) < 0$. *Then the extinction probability* $\{a_n\}$ *of the IBCP, starting from* $n \ge 1$, *possesses the following asymptotic behaviour,*

$$a_n \sim k_1 n^{-\alpha} \rho_c^n + k_2 n^{-\beta} \xi_c^n \quad (n \to \infty) \tag{3.17}$$

where k_1 *and* k_2 *are constants and* $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$ *and* $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$. *Furthermore, we have*

$$a_n \sim k \rho_c^n \cdot n^{-\alpha} \quad (n \to \infty) \tag{3.18}$$

where $-1 < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$ *and* k *is a constant.*

Proof. By Theorem 5.6 of Chen et al [4], we know that if $\rho_c < \rho_b \le 1$ and $C'(\rho_c)+2B(\rho_c) < 0$, then the extinction probability $\{a_n\}$ is given by

$$a_n = \frac{\int_{\xi_c}^{\rho_c} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy}{\int_{\xi_c}^{\rho_c} \frac{B'(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy}. \tag{3.19}$$

where

$$A_1(s) = \frac{C(s)B(s)}{2}, \quad B_1(s) = \frac{B(s)(2B(s) + C'(s) - C(s)B'(s))}{2}. \tag{3.20}$$

It follows from (3.19) that there exists a constant k which is independent of n such that

$$a_n = k \cdot \int_{\xi_c}^{\rho_c} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy. \tag{3.21}$$

In order to understand the asymptotic property of $\{a_n\}$ in (3.21), we first carefully consider the property of the function $\exp\left\{\int_0^y \frac{B_1(x)}{A_1(x)} \mathrm{d}x\right\}$ which is the key term in the expression (3.21). Denote

$$a_n^+ = k \cdot \int_0^{\rho_c} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy$$

$$\text{and} \qquad a_n^- = k \cdot \int_{\xi_c}^{0} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} e^{\int_0^y \frac{B_1(x)}{A_1(x)} dx} dy. \tag{3.22}$$

Following Chen et al [4], we denote $A_0(s) = \frac{C(s)}{2}$ and $B_0(s) = B(s)$, then $B_1(s)$ and $A_1(s)$ given in (3.20) can be rewritten as

$$A_1(s) = A_0(s)B_0(s)$$

and

$$B_1(s) = B_0(s)\left[B_0(s) + A_0'(s)\right] - A_0(s)B_0'(s).$$

Hence

$$\begin{aligned}\int_0^y \frac{B_1(x)}{A_1(x)}\mathrm{d}x &= \int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x + \int_0^y \frac{A_0'(x)}{A_0(x)}\mathrm{d}x - \int_0^y \frac{B_0'(x)}{B_0(x)}\mathrm{d}x \\ &= \int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x + \ln\frac{A_0(y)}{B_0(y)} + \ln\frac{B_0(0)}{A_0(0)}\end{aligned}$$

where $B_0(0) = b_0 > 0$ and $A_0(0) = \frac{c_0}{2} > 0$.

It follows that

$$\exp\left\{\int_0^y \frac{B_1(x)}{A_1(x)}\mathrm{d}x\right\} = k_1 \cdot \frac{A_0(y)}{B_0(y)} \exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x\right\} \tag{3.23}$$

where k_1 is a constant which is independent of y. Substituting (3.23) into (3.22) shows that there exists a constant, denoted by k again, which is independent of both y and n, such that

$$a_n^+ = k \cdot \int_0^{\rho_c} \frac{ny^{n-1}B_0(y) - y^n B_0'(y)}{(B_0(y))^2} \exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x\right\} \mathrm{d}y.$$

Since $\rho_c < \rho_b \le 1$ we know that $B_0(s) \equiv B(s)$ has no zero on $[0, \rho_c]$ and thus $1/B_0(s)$ is bounded on $[0, \rho_c]$. It follows from this crucial fact and the mean-valued theorem together with the simple facts that both $B_0(s)$ and $B_0'(s)$ are bounded function of $s \in [0, \rho_c]$, we know that there exist two constants k_1 and k_2 which are both independent of y and n such that

$$a_n^+ = k_1 n \int_0^{\rho_c} y^{n-1} \exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x\right\} \mathrm{d}y + k_2 \cdot \int_0^{\rho_c} y^n \exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x\right\} \mathrm{d}y.$$

However, the function $\exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x\right\}$ is just $A(y)$ defined in (2.6) and thus by using (ii) of Lemma 3.1 once again we know that there exist two constants, again denoted by k_1 and k_2, such that

$$a_n^+ = k_1 n \int_0^{\rho_c} y^{n-1} (\rho_c - y)^\alpha \mathrm{d}y + k_2 \cdot \int_0^{\rho_c} y^n (\rho_c - y)^\alpha \mathrm{d}y \tag{3.24}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$.

Noting that $C'(\rho_c) + 2B(\rho_c) < 0$ and that $C'(\rho_c) < 0$, we know $1 + \frac{2B(\rho_c)}{C'(\rho_c)} > 0$, and thus

$$-1 < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0. \tag{3.25}$$

Therefore, we can obtain that

$$\begin{aligned}\int_0^{\rho_c} y^{n-1}(\rho_c - y)^{\alpha}\,\mathrm{d}y &= \rho_c^n \rho_c^{\alpha} \int_0^1 x^{n-1}(1-x)^{\alpha}\mathrm{d}x \\ &= \rho_c^{n+\alpha} \cdot \int_0^1 x^{n-1}(1-x)^{1+\alpha-1}\mathrm{d}x \\ &= \rho_c^{n+\alpha} \cdot \frac{\Gamma(n)\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)}. \end{aligned} \tag{3.26}$$

Similarly, we have

$$\int_0^{\rho_c} y^{n}(\rho_c - y)^{\alpha}\,\mathrm{d}y = \rho_c^{n+1+\alpha} \cdot \frac{\Gamma(n+1)\Gamma(1+\alpha)}{\Gamma(n+2+\alpha)}. \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.24), using the fact that $1+\alpha > 0$, and applying Lemma 3.3 together with some trivial algebra we then can write that there exists a constant k such that

$$a_n^+ \sim k_1 n^{-\alpha} \rho_c^n \quad (n \to \infty).$$

Similarly we can get

$$a_n^- \sim k_2 n^{-\beta} \xi_c^n \quad (n \to \infty)$$

with $\beta = \frac{2B(\xi_c)}{C'(\xi_c)} > 0$. Then (3.17) follows. Again, (3.18) follows from (3.17) directly.

Finally we consider the subcase that $C'(\rho_c)+2B(\rho_c) > 0$. Since now $C'(\rho_c)+2B(\rho_c) > 0$, but $C'(\rho_c) < 0$ and $B(\rho_c) > 0$ we can certainly find the smallest positive integer $m \geq 2$ such that $mC'(\rho_c) + 2B(\rho_c) \leq 0$ but for all $1 \leq n < m$ we have $nC'(\rho_c) + 2B(\rho_c) > 0$. Equivalently, if we let $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$, then $0 < m-1 < -\alpha \leq m$. We first consider the easy subcase of $mC'(\rho_c) + 2B(\rho_c) = 0$.

Theorem 3.9. *Suppose $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) > 0$. If there exists a positive integer m such that $mC'(\rho_c)+2B(\rho_c) = 0$, then there exist $(m+1)$ constants $\{k_0, k_1, \cdots, k_m\}$ with $k_0 = 1$ such that the extinction probability $\{a_n\}$, starting from $n \geq 1$, can be written as*

$$a_n = \sum_{l=0}^{m} k_l n^l \rho_c^{n-l}. \tag{3.28}$$

In particular, there exists a constant k such that

$$a_n \sim k n^{-\alpha} \rho_c^n \quad (n \to \infty) \tag{3.29}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} = -m$.

Proof. Expression (3.28) follows from (5.23) in Theorem 5.7 of Chen et al [4] directly. Then (3.29) is an easy consequence of (3.28) by noting that we have denoted m as $-\alpha$.

Note that Theorem 3.7 can be viewed as a special case of Theorem 3.9 when $m = 1$.

We now turn to the final sub-case of $mC'(\rho_c) + 2B(\rho_c) < 0$ for some $m \geq 2$ where m is the smallest positive integer that makes $mC'(\rho_c) + 2B(\rho_c) < 0$ is true. Then, as detailed explained in Chen et al [4], in addition to define $A_0(s) = \frac{C(s)}{2}$ and $B_0(s) = B(s)$, we have to define $A_n(s)$ and $B_n(s)$ $(n \geq 1)$ sequentially as follows until we get $A_m(s)$ and $B_m(s)$.

$$A_n(s) = A_{n-1}(s)B_{n-1}(s) \tag{3.30}$$

$$B_n(s) = B_{n-1}(s)\left[B_{n-1}(s) + A'_{n-1}(s)\right] - A_{n-1}(s)B'_{n-1}(s). \tag{3.31}$$

As also detailed explained in Chen et al [4], without loss generality, we may assume that $A_m(s) > 0$ for all $s \in (\xi_c, \rho_c)$.

Theorem 3.10. *Suppose $\rho_c < \rho_b \leq 1$ and $C'(\rho_c)+2B(\rho_c) > 0$ and that $-\frac{2B(\rho_c)}{C'(\rho_c)}$ is not an integer. Let m be the smallest positive integer such that $m = \min\{k \geq 1, kC'(\rho_c) + 2B(\rho_c) < 0\}$ and thus $-m < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < -(m-1)$. Further assume that $A_m(s) > 0$ for all $s \in (\xi_c, \rho_c)$ where $A_m(s)$ is defined sequently as in (3.30) and (3.31). Then the extinction probability $\{a_n\}$ of the IBCP $(n \geq 1)$, starting from $n \geq 1$, possesses the asymptotic behaviour that there exist $(m+1)$ constants $\{k_0, k_1, \cdots, k_{m-1}\}$ such that*

$$a_n \sim \sum_{l=0}^{m} k_l \frac{n!}{(n-l)!} \rho_c^{n-l} n^{-\alpha} \quad (n \to \infty) \tag{3.32}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$. Furthermore, we have

$$a_n \sim k \cdot \rho_c^n n^{-\alpha} \quad (n \to \infty) \tag{3.33}$$

where $-m < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < -(m-1)$.

Proof. By Theorem 5.8 of Chen et al [4], we know that for sufficient large n, the extinction probability $\{a_n\}$ is given by

$$a_n = k \sum_{l=0}^{m} \frac{n!}{(n-l)!} \int_{\xi_c}^{\rho_c} \frac{y^{n-l} D_{m,l}(y)}{A_m(y)} \exp\left\{\int_0^y \frac{B_m(x)}{A_m(x)} \mathrm{d}x\right\} \mathrm{d}y \tag{3.34}$$

for some constant k that is independent of n where $A_m(s)$ and $B_m(s)$ are defined in (3.30) and (3.31) and the functions $D_{m,l}(s)$ etc are given recursively as

$$D_{1,0}(s) = -B'(s) \quad D_{1,1}(s) = B(s) \tag{3.35}$$

$$D_{n,k}(s) = D_{n-1,k-1}(s)B_{n-1}(s) - D_{n-1,k}(s)B'_{n-1}(s) + D'_{n-1,k}(s)B_{n-1}(s), \quad (k \leq n-1) \tag{3.36}$$

and

$$D_{n,n}(s) = \Pi_{m=0}^{n-1} B_m(s). \tag{3.37}$$

By (3.35) – (3.37), it is easily seen that all $D_{m,l}(s)$ are analytic function of s, since they are all power series of s. Hence they are all bounded on the finite interval $[\xi_c, \rho_c]$. It follows that the $\{a_n\}$ in (3.34) can be written as

$$a_n = \sum_{l=0}^{m} k_l \frac{n!}{(n-l)!} \int_{\xi_c}^{\rho_c} \frac{y^{n-l}}{A_m(y)} \exp\left\{\int_0^y \frac{B_m(x)}{A_m(x)} \mathrm{d}x\right\} \mathrm{d}y \tag{3.38}$$

where $\{k_0, k_1, \cdots, k_m\}$ are $(m+1)$ constants.

Again, let $\{a_n^+\}$ be the part of $\{a_n\}$ regarding the integral of "$\int_0^{\rho_c}$" and $\{a_n^-\}$ be the part regarding the integral of "$\int_{\xi_c}^0$" and thus $a_n = a_n^+ + a_n^-$.

Firstly, similarly as we get (3.23), we can get, by using (3.30) and (3.31), that

$$\frac{B_m(s)}{A_m(s)} = \frac{B_{m-1}(s)}{A_{m-1}(s)} + \frac{A'_{m-1}(s)}{A_{m-1}(s)} - \frac{B'_{m-1}(s)}{B_{m-1}(s)} \tag{3.39}$$

and thus

$$\exp\left\{\int_0^y \frac{B_m(x)}{A_m(x)}\mathrm{d}x\right\} = \exp\left\{\int_0^y \frac{B_{m-1}(x)}{A_{m-1}(x)}\mathrm{d}x\right\} \frac{A_{m-1}(y)}{B_{m-1}(y)} \cdot \frac{B_{m-1}(0)}{A_{m-1}(0)}. \tag{3.40}$$

By repeating using (3.39) and (3.40) and noting that $\frac{B_{m-1}(0)}{A_{m-1}(0)}$ is just a constant, we get that

$$\exp\left\{\int_0^y \frac{B_m(x)}{A_m(x)}\mathrm{d}x\right\} = k \cdot \exp\left\{\int_0^y \frac{B_0(x)}{A_0(x)}\mathrm{d}x\right\} \cdot \frac{\Pi_{l=0}^{m-1} A_l(y)}{\Pi_{l=0}^{m-1} B_l(y)} \tag{3.41}$$

where k is a constant.

By using (3.30) we may easily see that for any $n \geq 1$, $A_n(s) = A_0(s)\Pi_{k=0}^{n-1} B_k(s)$. Substituting this latter expression into (3.41) and then substituting the resulting (3.41) into (3.38), we obtain that there exists $(m+1)$ constants, again denoted by $\{k_0, k_1, \cdots, k_m\}$, such that

$$a_n^+ = \sum_{l=0}^m k_l \frac{n!}{(n-l)!} \int_0^{\rho_c} y^{n-l} \frac{A_0(y)\Pi_{k=0}^{m-1} A_k(y)}{(A_m(y))^2} \exp\left\{\int_0^y \frac{2B(x)}{C(x)}\mathrm{d}x\right\} \mathrm{d}y. \tag{3.42}$$

Now since m is the minimal k such that $kC'(\rho_c) + 2B(\rho_c) < 0$, and hence ρ_c is not a zero of the function $A_0(y)\Pi_{k=0}^{m-1} A_k(y)/(A_m(y))^2$. Thus by applying mean-valued theorem together with (ii) of Lemma 3.1 we see that $\{a_n^+\}$ in (3.42) can be written as

$$a_n^+ = \sum_{l=0}^m k_l \cdot \frac{n!}{(n-l)!} \int_0^{\rho_c} y^{n-l} (\rho_c - y)^{-\alpha} \mathrm{d}y \tag{3.43}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0$.

Similarly, we have

$$a_n^- = \sum_{l=0}^m \tilde{k}_l \cdot \frac{n!}{(n-l)!} \int_{\xi_c}^0 y^{n-l} (y - \xi_c)^{-\beta} \mathrm{d}y. \tag{3.44}$$

Using the same transformation as we did before together with applying Lemma 3.3 and using the fact that $|\xi_c| < \rho_c < 1$, we can similarly prove (3.32). Then (3.33) follows directly from (3.32).

Remark 3.11. If we carefully check the results obtained in Theorems 3.6 to 3.10, particular expressions (3.13), (3.16), (3.18), (3.29) and (3.33), we may see that if the IBC q-matrix Q is irregular, then the extinction probabilities $\{a_n\}$ always satisfy uniformly the asymptotic behaviour

$$a_n \sim kn^{-\alpha}\rho_c^n \quad (n \to \infty) \tag{3.45}$$

where $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$ and k is a constant which is independent of n. Hence, Theorem 2.5 is fully proved. We also note that the basic conclusions in Theorems 3.6 to 3.10 are nothing but special cases of (3.45) with the value α being of $\alpha > 0$ (Theorem 3.6), $\alpha = -1$ (Theorem 3.7), $-1 < \alpha < 0$ (Theorem 3.8), $\alpha = -m$ for some positive integer $m \geq 2$ (Theorem 3.9), and $-m < \alpha < -(m-1)$ for some positive integer $m \geq 2$ (Theorem 3.10), respectively.

Remark 3.12. By checking Remark 3.11, it seems that the case of $\alpha = 0$ is missing! Note that, however, in discussing irregular case we have omitted the case of $\rho_b = \rho_c < 1$ for its triviality. Now, we can see that even for $\rho_b = \rho_c < 1$, (3.45) is still true in the sense of $k = 1$ and $\alpha = 0$. Indeed, in this case we have $a_n = \rho_c^n$ and thus (3.45) takes the form of $\alpha = 0$. Thus this case fill the gap of the "spectrum" from minus infinity to positive infinity well distributed among Theorems 3.6 to 3.10.

Finally, we use a simple example that was discussed both in Kalinkin [10] and Chen et al [4] to end this paper. In this simple example the birth structure for both the branching and collision components takes a single birth form. More specifically, we assume

$$b_0 = a > 0, \quad b_1 = -(a+b), \quad b_2 = b > 0, \quad b_j \equiv 0 \quad (\forall j \geq 3) \tag{3.46}$$

and that

$$c_0 = d > 0, \quad c_1 = r \geq 0, \quad c_2 = -(d+r+c), \quad c_3 = c > 0, \quad c_j \equiv 0 \quad (\forall j \geq 4). \tag{3.47}$$

Using the above quantities, we can easily construct an IBC q-matrix Q. It is clear that for this IBC q-matrix Q, we have

$$B(s) = a - (a+b)s + bs^2 = a(1-s)(1-\frac{bs}{a}) \tag{3.48}$$

and

$$C(s) = d + rs - (d+r+c)s^2 + cs^3 = c(s-1)(s-\rho_c)(s-\xi_c) \tag{3.49}$$

where $\rho_c = \frac{(d+r)+\sqrt{(d+r)^2+4dc}}{2c}$ and $\xi_c = \frac{(d+r)-\sqrt{(d+r)^2+4dc}}{2c} < 0$. It is easily seen that $C'(1) = c - (2d+r)$ and $B'(1) = b - a$.

By Theorem 6.1 in Chen et al [4], we know that for this IBC-q-matrix Q, the extinction probabilities are less then 1 if and only if either $c > 2d + r$, or $c = (2d+r)$, $b > a$ and $3d + r < 2(b-a)$. Hence in studying asymptotic behavior, we only need to consider these latter two cases. Now combining our Theorems 2.4 and 2.5 and Theorem 6.1 in Chen et al [4], we can get the following satisfactory conclusion.

Corollary 3.1. *For the IBC-q-matrix determined by (3.46) and (3.47) we have the following conclusions.*

(i) There always exists only one IBCP which is the Feller minimal process and that this Feller minimal process is honest if and only if $c \leq 2d + r$.

(ii) The extinction probabilities $a_n = 1$ $(\forall n \geq 1)$ *if and only if one of the following three cases holds*

(a) $c < (2d + r)$,

(b) $c = (2d + r)$ *and* $b \leq a$,

(c) $c = (2d + r)$, $b > a$ *and* $3d + r \geq 2(b - a)$.

(iii) If $c = (2d + r)$, $b > a$ *and* $3d + r < 2(b - a)$, *then* $a_n < 1$ $(\forall n \geq 1)$ *and in this case, the asymptotic behaviour of the extinction probability* $\{a_n\}$ *is given by*

$$a_n \sim kn^{-\alpha} \quad (n \to \infty)$$

where $\alpha = \frac{2(b-a)}{3d+r} - 1 > 0$.

(iv) If $c > 2d + r$, *then* $a_n < 1$ $(\forall n \geq 1)$ *and in this case, the asymptotic behaviour of the extinction probabilities follows the power law of* $a_n \sim kn^{-\alpha}\rho_c^n$ $(n \to \infty)$ *where* ρ_c *is given below the expression (3.49),* $\alpha = \frac{2B(\rho_c)}{C'(\rho_c)}$ *which is easily given, and* k *is a constant.*

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